Uncertainty Measures of Roughness of Knowledge and Rough Sets in Ordered Information Systems

Wei-Hua Xu[1], Hong-zhi Yang[2], and Wen-Xiu Zhang[3]

[1] Institute for Information and System Sciences,
Xi'an Jiaotong University, Xi'an, Shaan'xi 710049, P.R. China
datongxuweihua@126.com
[2] He'nan Pingyuan University, Xinxiang 453003, P.R. China
Institute for Information and System Sciences,
Xi'an Jiaotong University, Xi'an, Shaan'xi 710049, P.R. China
yzxz@163.com
[3] Faculty of Science, Institute for Information and System Sciences,
Xi'an Jiaotong University, Xi'an, Shaan'xi 710049, P.R. China
wxzhang@mail.xjtu.edu.cn

Abstract. Rough set theory has been considered as a useful tool to deal with inexact, uncertain, or vague knowledge. However, in real-world, most of information systems are based on dominance relations, called ordered information systems, in stead of the classical equivalence for various factors. So, it is necessary to find a new measure to knowledge and rough set in ordered information systems. In this paper, we address uncertainty measures of roughness of knowledge and rough sets by introducing rough entropy in ordered information systems. We prove that the rough entropy of knowledge and rough set decreases monotonously as the granularity of information becomes finer, and obtain some conclusions, which is every helpful in future research works of ordered information systems.

Keywords: Rough set, Information systems, Dominance relation, Rough entropy, Rough degree.

1 Introduction

The rough set theory, proposed by Pawlak in the early 1980s[1], is an extension of the classical set theory for modeling uncertainty or imprecision information. The research has recently roused great interest in the theoretical and application fronts, such as machine learning, pattern recognition, data analysis, and so on [2-6].

In Pawlak's original rough set theory, partition or equivalence (indiscernibility relation) is a important and primitive concept. However, partition or equivalence relation, as the indiscernibility relation in Pawlak's original rough set theory, is still restrictive for many applications. To address this issue, several interesting

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and meaningful extensions to equivalence relation have been proposed in the past, such as tolerance relations [17], similarity relations [16], others [18-20]. Particularly, Greco, Matarazzo, and Slowinski[7-11] proposed an extension rough sets theory, called the dominance-based rough sets approach(DRSA) to take into account the ordering properties of attributes. This innovation is mainly based on substitution of the indiscernibility relation by a dominance relation. In DRSA condition attributes and classes are preference ordered. And many studies have been made in DRSA[12-15].

On the other hand, the concept of entropy, originally defined by Shannon in 1948 for communication theory, gives a measure of uncertainty about the structure of a system. It has been useful concept for characterizing information content in a great diversity of models and applications. Attempts have been made to use Shannon's entropy to measure uncertainty in rough set theory [21-24]. Moreover, information entropy is introduced into incomplete information systems, and a kind of new rough entropy is defined to describe the incomplete information systems and roughness of rough set. While, most of information systems are based on dominance relations, i.e., ordered information systems. Hence, consideration of the uncertain measure about entropy in ordered information systems is needed. This paper discussed the problem mainly.

In this paper, we address uncertainty measures of roughness of knowledge and rough sets by introducing rough entropy in ordered information systems. We prove that the rough entropy of knowledge and rough set decreases monotonously as the granularity of information becomes finer, and obtain some conclusions, which is every helpful in future research works of ordered information systems.

2 Rough Sets and Ordered Information Systems

The following recalls necessary concepts and preliminaries required in the sequel of our work. Detailed description of the theory can be found in [4,15].

The notion of information system (sometimes called data tables, attribute-value systems, knowledge representation systems etc.) provides a convenient tool for the representation of objects in terms of their attribute values.

An information system is an ordered quadruple $\mathcal{I} = (U, A, F)$, where $U = \{x_1, x_2, \cdots, x_n\}$ is a non-empty finite set of objects called the universe, and $A = \{a_1, a_2, \cdots, a_p\}$ is a non-empty finite set of attributes, such that there exists a map $f_l : U \to V_{a_l}$ for any $a_l \in A$, where V_{a_l} is called the domain of the attribute a_l, and denoted $F = \{f_l | a_l \in A\}$.

In an information systems, if the domain of a attribute is ordered according to a decreasing or increasing preference, then the attribute is a criterion.

Definition 2.1. An information system is called an ordered information system(OIS) if all condition attributes are criterions.

Assumed that the domain of a criterion $a \in A$ is complete pre-ordered by an outranking relation $\succeq_a$, then $x \succeq_a y$ means that x is at least as good as y with respect to criterion a. And we can say that x dominates y. In the following, without any loss of generality, we consider criterions having a numerical domain, that is, $V_a \subseteq \mathcal{R}$($\mathcal{R}$ denotes the set of real numbers).

We define $x \succeq y$ by $f(x,a) \geq f(y,a)$ according to increasing preference, where $a \in A$ and $x, y \in U$. For a subset of attributes $B \subseteq A$, $x \succeq_B y$ means that $x \succeq_a y$ for any $a \in B$, and that is to say x dominates y with respect to all attributes in B. Furthermore, we denote $x \succeq_B y$ by $xR_B^{\geq}y$. In general, we denote a ordered information systems by $\mathcal{I}^{\succeq} = (U, A, F)$. Thus the following definition can be obtained.

Definition 2.2. Let $\mathcal{I}^{\succeq} = (U, A, F)$ be an ordered information, for $B \subseteq A$, denote

$$R_B^{\succeq} = \{(x,y) \in U \times U | f_l(x) \geq f_l(y), \forall a_l \in B\};$$

$R_B^{\succeq}$ are called dominance relations of ordered information system $\mathcal{I}^{\succeq}$.

Let denote

$$\begin{aligned} [x_i]_B^{\succeq} &= \{x_j \in U | (x_j, x_i) \in R_B^{\succeq}\} \\ &= \{x_j \in U | f_l(x_j) \geq f_l(x_i), \forall a_l \in B\}; \\ U/R_B^{\succeq} &= \{[x_i]_B^{\succeq} | x_i \in U\}, \end{aligned}$$

where $i \in \{1, 2, \cdots, |U|\}$, then $[x_i]_B^{\succeq}$ will be called a dominance class or the granularity of information, and $U/R_B^{\succeq}$ be called a classification of U about attribute set B.

The following properties of a dominance relation are trivial by the above definition.

Proposition 2.1. Let $R_A^{\succeq}$ be a dominance relation.

(1) $R_A^{\succeq}$ is reflexive,transitive, but not symmetric, so it is not a equivalence relation.

(2) If $B \subseteq A$, then $R_A^{\succeq} \subseteq R_B^{\succeq}$.

(3) If $B \subseteq A$, then $[x_i]_A^{\succeq} \subseteq [x_i]_B^{\succeq}$

(4) If $x_j \in [x_i]_A^{\succeq}$, then $[x_j]_A^{\succeq} \subseteq [x_i]_A^{\succeq}$ and $[x_i]_A^{\succeq} = \cup\{[x_j]_A^{\succeq} | x_j \in [x_i]_A^{\succeq}\}$.

(5) $[x_j]_A^{\succeq} = [x_i]_A^{\succeq}$ iff $f(x_i, a) = f(x_j, a)$ $(\forall a \in A)$.

(6) $|[x_i]_B^{\succeq}| \geq 1$ for any $x_i \in U$.

(7) $U/R_B^{\succeq}$ constitute a covering of U, i.e., for every $x \in U$ we have that $[x]_B^{\succeq} \neq \phi$ and $\bigcup_{x \in U} [x]_B^{\succeq} = U$.

where $|\cdot|$ denote cardinality of the set.

For any subset X of U, and A of $\mathcal{I}^{\succeq}$ define

$$\underline{R_A^{\succeq}}(X) = \{x \in U | [x]_A^{\succeq} \subseteq X\}; \quad \overline{R_A^{\succeq}}(X) = \{x \in U | [x]_A^{\succeq} \cap X \neq \phi\},$$

$\underline{R_A^{\succeq}}(X)$ and $\overline{R_A^{\succeq}}(x)$ are said to be the lower and upper approximation of X with respect to a dominance relation $R_A^{\succeq}$. And the approximations have also some properties which are similar to those of Pawlak approximation spaces.

Proposition 2.2. Let $\mathcal{I}^{\succeq} = (U, A, F)$ be an ordered information systems and $X, Y \subseteq U$, then its lower and upper approximations satisfy the following properties.

(1) $\underline{R_A^{\succeq}}(X) \subseteq X \subseteq \overline{R_A^{\succeq}}(X)$.

(2) $\overline{R_A^{\succeq}}(X \cup Y) = \overline{R_A^{\succeq}}(X) \cup \overline{R_A^{\succeq}}(Y)$;
$\underline{R_A^{\succeq}}(X \cap Y) = \underline{R_A^{\succeq}}(X) \cap \underline{R_A^{\succeq}}(Y)$.

(3) $\underline{R_A^{\succeq}}(X) \cup \underline{R_A^{\succeq}}(Y) \subseteq \underline{R_A^{\succeq}}(X \cup Y)$;
$\overline{R_A^{\succeq}}(X \cap Y) \subseteq \overline{R_A^{\succeq}}(X) \cap \overline{R_A^{\succeq}}(Y)$.

(4) $\underline{R_X^{\succeq}}(\sim X) = \sim \overline{R_A^{\succeq}}(X)$; $\overline{R_A^{\succeq}}(\sim X) = \sim \underline{R_A^{\succeq}}(X)$.

(5) $\underline{R_A^{\succeq}}(U) = U$; $\overline{R_A^{\succeq}}(\phi) = \phi$.

(6) $\underline{R_A^{\succeq}}(X) \subseteq \underline{R_A^{\succeq}}(\underline{R_A^{\succeq}}(X))$; $\overline{R_A^{\succeq}}(\overline{R_A^{\succeq}}(X)) \subseteq \overline{R_A^{\succeq}}(X)$.

(7) If $X \subseteq Y$, then $\underline{R_A^{\succeq}}(X) \subseteq \underline{R_A^{\succeq}}(Y)$ and $\overline{R_A^{\succeq}}(X) \subseteq \overline{R_A^{\succeq}}(Y)$.

where $\sim X$ is the complement of X.

Definition 2.3. For a ordered information system $\mathcal{I}^{\succeq} = (U, A, F)$ and $B, C \subseteq A$.

(1) If $[x]_B^{\succeq} = [x]_C^{\succeq}$ for any $x \in U$, then we call that classification $U/R_B^{\succeq}$ is equal to $R/R_C^{\succeq}$, denoted by $U/R_B^{\succeq} = U/R_C^{\succeq}$.

(2) If $[x]_B^{\succeq} \subseteq [x]_C^{\succeq}$ for any $x \in U$, then we call that classification $U/R_B^{\succeq}$ is finer than $R/R_C^{\succeq}$, denoted by $U/R_B^{\succeq} \subseteq U/R_C^{\succeq}$.

(3) If $[x]_B^{\succeq} \subseteq [x]_C^{\succeq}$ for any $x \in U$ and $[x]_B^{\succeq} \neq [x]_C^{\succeq}$ for some $x \in U$, then we call that classification $U/R_B^{\succeq}$ is properly finer then $R/R_C^{\succeq}$, denoted by $U/R_B^{\succeq} \subset U/R_C^{\succeq}$.

For a ordered information system $\mathcal{I}^{\succeq} = (U, A, F)$ and $B \subseteq A$, it is obtained that $U/R_A^{\succeq} \subseteq U/R_B^{\succeq}$ directly by Proposition 2.1(3) and above definition. And an ordered information systems $\mathcal{I}^{\succeq} = (U, A, F)$ be regarded as knowledge representation system $U/R_A^{\succeq}$ or knowledge A, as is same to classical rough set based on equivalence relation.

Example 2.1. Given an ordered information system in Table 1.

Table 1 An ordered information system

$U \times A$	a_1	a_2	a_3
x_1	1	2	1
x_2	3	2	2
x_3	1	1	2
x_4	2	1	3
x_5	3	3	2
x_6	3	2	3

If denote $B = \{a_1, a_2\}$, from the table we have

$$[x_1]_A^{\succeq} = \{x_1, x_2, x_5, x_6\};$$

$$[x_2]_A^{\succeq} = \{x_2, x_5, x_6\};$$
$$[x_3]_A^{\succeq} = \{x_2, x_3, x_4, x_5, x_6\};$$
$$[x_4]_A^{\succeq} = \{x_4, x_6\};$$
$$[x_5]_A^{\succeq} = \{x_5\};$$
$$[x_6]_A^{\succeq} = \{x_6\};$$

and

$$[x_1]_B^{\succeq} = \{x_1, x_2, x_5, x_6\};$$
$$[x_2]_B^{\succeq} = \{x_2, x_5, x_6\};$$
$$[x_3]_B^{\succeq} = \{x_1, x_2, x_3, x_4, x_5, x_6\};$$
$$[x_4]_B^{\succeq} = \{x_2, x_4, x_5, x_6\};$$
$$[x_5]_B^{\succeq} = \{x_5\};$$
$$[x_6]_B^{\succeq} = \{x_5, x_6\}.$$

Thus, it is obviously that $U/R_A^{\succeq} \subseteq U/R_B^{\succeq}$, i.e., classification $U/R_A^{\succeq}$ is finer than classification $U/R_B^{\succeq}$.

For simple description, in the following information systems are based on dominance relations generally, i.e., ordered information systems.

3 Rough Entropy of Knowledge in Ordered Information Systems

In classical rough set theory, knowledge be regarded as partition of set of objects to an information system. However, it is known that equality relations is replaced by dominance relations in an ordered information system. Thus, knowledge be regarded as classification of set of objects to an ordered information system by section 2.

In this section, we will introduce rough entropy of knowledge and establish relationships between roughness of knowledge and rough entropy in ordered information systems.

Firstly, let give the concept of rough entropy of knowledge in ordered information systems.

Definition 3.1. Let $\mathcal{I}^{\succeq} = (U, A, F)$ be an ordered information systems and $B \subseteq A$. The rough entropy of knowledge B is defined as follows:

$$E(B) = \sum_{i=1}^{|U|} \frac{|[x_i]_B^{\succeq}|}{|U|} \cdot \log_2 |[x_i]_B^{\succeq}| \ ,$$

where $|\cdot|$ is the cardinality of sets.

Example 3.1. In Example 2.1, the rough entropy of knowledge $A = \{a_1, a_2, a_3\}$ can be calculated by above definition, which is

$$\begin{aligned} E(A) &= \frac{4}{6}\cdot\log_2 4 + \frac{3}{6}\cdot\log_2 3 + \frac{5}{6}\cdot\log_2 5 + \\ &\quad \frac{2}{6}\cdot\log_2 2 + \frac{1}{6}\cdot\log_2 1 + \frac{1}{6}\cdot\log_2 1 \\ &= \frac{2}{3}\cdot 2 + \frac{1}{2}\cdot\log_2 3 + \frac{5}{6}\cdot\log_2 5 + \frac{1}{3} \\ &= 4.39409 \end{aligned}$$

Proposition 3.1. Let $\mathcal{I}^{\succeq} = (U, A, F)$ be an ordered information systems and $B \subseteq A$. The following hold.

(1) $E(B)$ can obtain its maximum $|U|\cdot\log_2|U|$, iff $U/R_B^{\succeq} = U$.
(2) $E(B)$ can obtain its minimum 0, iff $U/R_B^{\succeq} = \{\{x_1\}, \{x_2\}, \cdots, \{x_{|U|}\}\}$.

Proof. It is straightforward by Definition 3.1.

From Proposition 3.1, it can be concluded that information quantity provided by knowledge B is zero when its rough entropy reaches maximum, and its cannot distinguish any two objects in U, when the classification of ordered information systems is no meaning. When the rough entropy of knowledge B obtains its minimum, the information quantity is the most and every objects can be discriminated by B in the ordered information systems.

Theorem 3.1. Let $\mathcal{I}^{\succeq} = (U, A, F)$ be an ordered information systems and $B_1, B_2 \subseteq A$. If $U/R_{B_1}^{\succeq} \subset U/R_{B_2}^{\succeq}$, then $E(B_1) < E(B_2)$.

Proof. Because of $U/R_{B_1}^{\succeq} \subset U/R_{B_2}^{\succeq}$, we have that $[x_i]_{B_1}^{\succeq} \subseteq [x_i]_{B_2}^{\succeq}$ for every $x_i \in U$. Thus there exists some $x_j \in U$ such that $|[x_j]_{B_1}^{\succeq}| < |[x_j]_{B_2}^{\succeq}|$. Hence, by the Proposition 2.1 and Definition 3.1 we can obtain

$$\sum_{i=1}^{|U|} |[x_i]_{B_1}^{\succeq}|\cdot\log_2|[x_i]_{B_1}^{\succeq}| < \sum_{i=1}^{|U|} |[x_i]_{B_2}^{\succeq}|\cdot\log_2|[x_i]_{B_2}^{\succeq}|,$$

i.e.,

$$E(B_1) < E(B_2).$$

From Theorem 3.1, we can find that rough entropy of knowledge decreased monotonously as the granularity of information became smaller through finer classifications of objects set U.

Corollary 3.1. Let $\mathcal{I}^{\succeq} = (U, A, F)$ be an ordered information systems and $B_1, B_2 \subseteq A$. If $B_2 \subseteq B_1$, then $E(B_1) \leq E(B_2)$.

Theorem 3.2. Let $\mathcal{I}^{\succeq} = (U, A, F)$ be an ordered information systems and $B_1, B_2 \subseteq A$. If $U/R_{B_1}^{\succeq} = U/R_{B_2}^{\succeq}$, then $E(B_1) = E(B_2)$.

Proof. Since $U/R_{B_1}^{\succeq} = U/R_{B_2}^{\succeq}$, we have that $[x_i]_{B_1}^{\succeq} = [x_i]_{B_2}^{\succeq}$ for every $x_i \in U$. Thus, it is obtain $E(B_1) = E(B_2)$ directly.

The theorem states that two equivalence knowledge representation systems have same rough entropy.

Theorem 3.3. Let $\mathcal{I}^{\succeq} = (U, A, F)$ be an ordered information systems and $B_1, B_2 \subseteq A$. If $U/R^{\succeq}_{B_1} \subseteq U/R^{\succeq}_{B_2}$ and $E(B_1) = E(B_2)$, then $U/R^{\succeq}_{B_1} = U/R^{\succeq}_{B_2}$.

Proof. Since $E(B_1) = E(B_2)$, it follows that

$$\sum_{i=1}^{|U|} |[x_i]^{\succeq}_{B_1}| \cdot \log_2 |[x_i]^{\succeq}_{B_1}| = \sum_{i=1}^{|U|} |[x_i]^{\succeq}_{B_2}| \cdot \log_2 |[x_i]^{\succeq}_{B_2}|. \qquad (*)$$

From $U/R^{\succeq}_{B_1} \subseteq U/R^{\succeq}_{B_2}$, we have that $[x_i]^{\succeq}_{B_1} \subseteq [x_i]^{\succeq}_{B_2}$ for every $x_i \in U$. This show that $1 \leq |[x_i]^{\succeq}_{B_1}| \leq |[x_i]^{\succeq}_{B_2}|$. Thus, it is true that

$$|[x_i]^{\succeq}_{B_1}| \cdot \log_2 |[x_i]^{\succeq}_{B_1}| \leq |[x_i]^{\succeq}_{B_2}| \cdot \log_2 |[x_i]^{\succeq}_{B_2}|.$$

By the formula $(*)$, it follows that

$$|[x_i]^{\succeq}_{B_1}| \cdot \log_2 |[x_i]^{\succeq}_{B_1}| = |[x_i]^{\succeq}_{B_2}| \cdot \log_2 |[x_i]^{\succeq}_{B_2}|.$$

So, we easily obtain $|[x_i]^{\succeq}_{B_1}| = |[x_i]^{\succeq}_{B_2}|$, for every $x_i \in U$.

On the other hand, $[x_i]^{\succeq}_{B_1} \subseteq [x_i]^{\succeq}_{B_2}$, we get $[x_i]^{\succeq}_{B_1} = [x_i]^{\succeq}_{B_2}$ for every $x_i \in U$.

Hence, $U/R^{\succeq}_{B_1} = U/R^{\succeq}_{B_2}$.

Theorem 3.3 states that if two knowledge representation systems exists inclusion relation and their rough entropy are equal, then two knowledge representation systems is equivalent.

Corollary 3.2. Let $\mathcal{I}^{\succeq} = (U, A, F)$ be an ordered information systems and $B_1, B_2 \subseteq A$. If $B_2 \subseteq B_1$ and $E(B_1) = E(B_2)$, then $U/R^{\succeq}_{B_1} = U/R^{\succeq}_{B_2}$.

Example 3.2. We had got that $U/R^{\succeq}_{A} \subseteq U/R^{\succeq}_{B}$, if denote $B = \{a_1, a_2\}$ in the ordered information system of Example 2.1. Moreover, $E(B)$ cab be calculated easily, which is

$$\begin{aligned} E(B) &= \frac{4}{6} \cdot \log_2 4 + \frac{3}{6} \cdot \log_2 3 + \frac{6}{6} \cdot \log_2 6 + \\ &\quad \frac{4}{6} \cdot \log_2 4 + \frac{1}{6} \cdot \log_2 1 + \frac{2}{6} \cdot \log_2 2 \\ &= \frac{2}{3} \cdot 4 + \frac{1}{2} \cdot \log_2 3 + \log_2 6 + \frac{1}{3} \\ &= 6.37744 \end{aligned}$$

On the other hand, by Example 3.1, we obtained $E(A) = 4.39409$.

Thus, it is obvious that $E(A) \leq E(B)$.

However, if denote $B' = \{a_1\}$ and $B'' = \{a_2\}$ in the system of Example 2.1, we have that

$$[x_1]^{\succeq}_{B'} = [x_3]^{\succeq}_{B'} = \{x_1, x_2, x_3, x_4, x_5, x_6\};$$

$$[x_2]^{\succeq}_{B'} = [x_5]^{\succeq}_{B'} = [x_6]^{\succeq}_{B'} = \{x_2, x_5, x_6\};$$
$$[x_4]^{\succeq}_{B'} = \{x_2, x_4, x_5, x_6\},$$

and

$$[x_1]^{\succeq}_{B''} = [x_2]^{\succeq}_{B''} = [x_6]^{\succeq}_{B''} = \{x_1, x_2, x_5, x_6\};$$
$$[x_3]^{\succeq}_{B''} = [x_4]^{\succeq}_{B''} = \{x_1, x_2, x_3, x_4, x_5, x_6\};$$
$$[x_5]^{\succeq}_{B''} = \{x_5\}.$$

Furthermore, we can obtain that $E(B') = 8.88071$ and $E(B'') = 9.16993$, which show $E(B') < E(B'')$. While, $U/R^{\succeq}_{B'} \subseteq U/R^{\succeq}_{B''}$ doesn't hold. So, it can be concluded that the converse proposition of Theorem 3.1 does not hold.

4 Rough Entropy of Rough Sets in Ordered Information Systems

In rough set theory, the roughness of a rough set can be measured by its rough degree. So we give the rough degree of a rough set in ordered information systems.

Definition 4.1. Let $\mathcal{I}^{\succeq} = (U, A, F)$ be an ordered information systems and $B \subseteq A$. The rough degree of a rough set $X \subseteq U$ about knowledge B is defined as follows:

$$\rho_B(X) = 1 - \frac{|\underline{R^{\succeq}_B}(X)|}{|\overline{R^{\succeq}_B}(X)|},$$

where $|\cdot|$ is the cardinality of sets.

From the above definition and Proposition 2.2, it is obvious to $0 \leq \rho_B(X) \leq 1$, and the following property can be obtained easily.

Theorem 4.1. Let $\mathcal{I}^{\succeq} = (U, A, F)$ be an ordered information systems and $B_1, B_2 \subseteq A$. If $U/R^{\succeq}_{B_1} \subseteq U/R^{\succeq}_{B_2}$, then $\rho_{B_1}(X) \leq \rho_{B_2}(X)$, for any rough set $X \subseteq U$.

Example 4.1. In Example 2.1, we have known $U/R^{\succeq}_A \subseteq U/R^{\succeq}_B$, i.e., classification $U/R^{\succeq}_A$ is finer than classification $U/R^{\succeq}_B$ in the system of Table 1.

For $X = \{x_4, x_5, x_6\}$, we have

$$\underline{R^{\succeq}_A}(X) = \{x_4, x_5, x_6\}, \quad \overline{R^{\succeq}_A}(X) = U;$$
$$\underline{R^{\succeq}_B}(X) = \{x_5, x_6\}, \quad \overline{R^{\succeq}_B}(X) = U.$$

Thus, by calculating, the rough degrees of X about knowledge B and A can be obtained respectively, which are

$$\rho_A(X) = \frac{1}{2}; \quad \rho_B(X) = \frac{2}{3};$$

Obviously, $\rho_A(X) \leq \rho_B(X)$.

From Theorem 4.1 and Example 4.1, we can get that coarser is the classification of ordered information systems, smaller is not the rough degree of a rough set of the system.

However, it can be find that the uncertainty measure, i.e., rough degree, of a rough set is not exact in ordered information systems by the following example.

Example 4.2. Let $X' = \{x_3, x_5, x_6\}$ in Example 4.1, we get

$$\underline{R_A^{\succeq}}(X') = \underline{R_B^{\succeq}}(X') = \{x_5, x_6\};$$
$$\overline{R_A^{\succeq}}(X') = \overline{R_B^{\succeq}}(X') = U.$$

So have

$$\rho_A(X) = \rho_B(X) = \frac{1}{3}\ .$$

In other words, the uncertainty of knowledge B is larger than that of A in Example 4.2, but X' has the same rough degree. Therefore, it is necessary to find a new and more accurate uncertainty measure for rough sets in ordered information systems.

Definition 4.2. Let $\mathcal{I}^{\succeq} = (U, A, F)$ be an ordered information systems and $B \subseteq A$. The rough entropy of a rough set $X \subseteq U$ about knowledge B is defined as follows:

$$E_B(X) = \rho_B(X)E(B).$$

From Definition 4.2, the rough entropy of rough sets is related not only to its own rough degree, but also to the uncertainty of knowledge in the ordered information systems.

Example 4.3. The rough entropy of X' in Example 4.2 is calculated about knowledge B and A respectively, which are

$$E_B(X') = \rho(X')E(B) = \frac{1}{3} \times 6.37744 = 2.12579;$$
$$E_A(X') = \rho(X')E(A) = \frac{1}{3} \times 4.39409 = 1.46468.$$

Thus, we have

$$E_A(X') < E_B(X').$$

By this example, it is obvious that the rough entropy of rough sets is more accurate than the rough degree to measure the roughness of rough sets in ordered information systems.

Furthermore, the following property can be obtained about the entropy of rough sets.

Theorem 4.2. Let $\mathcal{I}^{\succeq} = (U, A, F)$ be an ordered information systems and $B_1, B_2 \subseteq A$. If $U/R_{B_1}^{\succeq} \subset U/R_{B_2}^{\succeq}$, then $E_{B_1}(X) < E_{B_2}(X)$, for any $X \subseteq U$.

Proof. It is straightforward by Theorem 3.1 and Theorem 4.1.

Corollary 4.1. Let $\mathcal{I}^{\succeq} = (U, A, F)$ be an ordered information systems and $B_1, B_2 \subseteq A$. If $B_2 \subseteq B_1$, then $E_{B_1}(X) \leq E_{B_2}(X)$ for any $X \subseteq U$.

It can be deduced from the above propositions that the rough entropy of a rough set monotonously decreases as the classification becomes finer in ordered information systems.

5 Conclusions

Rough set theory is a new mathematical tool to deal with vagueness and uncertainty. Development of a rough computational method is one of the most important research tasks. While, in practise, ordered information system confines the applications of classical rough set theory. In this article, a measure to knowledge and its important properties are established by proposed rough entropy in ordered information systems. We prove that the rough entropy of knowledge and rough set decreases monotonously as the granularity of information becomes finer, and obtain some conclusions, which is every helpful in future research works of ordered information systems.

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