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ORIGINAL ARTICLE

Fuzzy rough set models over two universes

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Abstract The extension of rough set model is an important research direction in rough set theory. The aim of this paper is to present new extensions of the rough set model over two different universes which are rough fuzzy set model in a generalized approximation space, rough set model in a fuzzy approximation space and rough fuzzy set model in a fuzzy approximation space based over two different universes. Moreover, the properties of the approximation operators in these models are investigated. Furthermore, by employing cut set of fuzzy set and fuzzy relation, classical representations of fuzzy rough approximation operators are studied. Finally, the measures of fuzzy rough set models are presented, and the relationships among the fuzzy rough models and rough set model over two universes are investigated.

Keywords Rough set · Two universes · Fuzzy relation · Fuzzy approximation space · Measure

1 Introduction

Rough set theory was proposed by Pawlak [15–17] in 1982, has been successfully applied in the fields of artificial intelligence, pattern recognition, medical diagnosis, data mining, conflict analysis, algebra [1, 2, 4, 5, 9, 12, 18, 19, 20] and so on. In recent decades, the rough set theory has generated a great deal of interest among more and more researchers.

It is widely acknowledged that the theory of rough sets, which is important to construct a pair of upper and lower approximation operators, is based on available information. In the Pawlak approximation space, an arbitrary subset of the universe of discourse can be approximated by the lower and upper approximation sets. The lower approximation is the union of all equivalence classes which are generalized by the equivalence relation on the universe included in the given set, and the upper approximation is the union of all equivalence classes which are generalized by the equivalence relation on the universe having a nonempty intersection with the given set. So the equivalence relation is a key notion in Pawlak's rough set model.

However, the requirement of an equivalence relation on a universe seems to be a very restrictive condition, so it limits the applications of rough set theory. Therefore, some researchers have extended the Pawlak's rough set model by the other binary relations. For example, the notions of approximation operators have been generalized by tolerance relations [8, 13, 14] or similarity relations [36], dominance relations [30], and general approximation spaces. On the other hand, the rough sets in the fuzzy environment [3] and intuitionistic fuzzy environment [34] have become a rapidly progressing research area and have received much attention since Dubois and Prade firstly proposed the notions of rough fuzzy set and fuzzy rough set. Particular studies on fuzzy rough sets and rough fuzzy sets can be found in the literature [6, 7, 10, 12, 25,26, 27, 28, 29, 31, 33, 37, 38].

Moreover, the first study on the rough set model over two universes was done in 1995, and was one of the hottest

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researches in recent years. Shen et al. [23] researched the variable precision rough set model over two universes and investigated the properties, Yan et al. [32] studied on the model of rough set over dual-universe. More details about recent advancements of rough set model over two universes can be found in the literatures [10, 11, 21, 22, 24, 35]. In this paper, we will discuss the fuzzy rough set models over two universes in the fuzzy environment.

The structure of this paper is organized as follows: we briefly introduce necessary notions relevant to the present research in Sect. 2. In Sect. 3, we define some types of approximation operators in the generalized and fuzzy approximation space, respectively. All kinds of lower and upper approximation operators on α-level have been defined and the properties are investigated. In Sect. 4, the measures of fuzzy rough set models on different universes are researched. In Sect. 5, the relationships among fuzzy rough set models and rough set model over two universes are investigated. Finally, a brief conclusion is made in Sect. 6.

2 Preliminaries

The following recalls necessary concepts and preliminaries required in the sequel of our work. For one thing, we will give some related definitions of fuzzy set theory.

2.1 Fuzzy set and fuzzy relation

Let U be a finite and nonempty set called the universe of discourse.

A set A is said to be a fuzzy set if it is a mapping from U into the unit interval [0, 1]:

$$\mu_A : U \mapsto [0, 1],$$

where we call $\mu_A(x)$ is the membership degree of x in A.

The classes of all subsets (respectively, fuzzy subsets) of U will be denoted by $P(U)$ [respectively, by $F(U)$].

The α cut and strong α cut of A will be denoted by A_α and A_{α_+} as follows:

$$A_\alpha = \{x|\mu_A(x) \geq \alpha\}, \quad A_{\alpha_+} = \{x|\mu_A(x) > \alpha\},$$

where $\alpha \in [0, 1]$.

Especially, if $\alpha = 0$, then $A_0 = U$; if $\alpha = 1$, then $A_{1_+} = \emptyset$.

Definition 2.1 Let U and V be two finite and nonempty universes, $R \in P(U \times V)$ be called as a binary relation from U to V.

If $U = V$, R is referred to a binary relation on $U \times U$.

Let R be a binary relation on $U \times U$. R is a reflexive relation, if for any $x \in U$, we have $(x, x) \in R; R$ is a symmetric relation, if for any $x, y \in U, (x, y) \in R \Rightarrow (y, x) \in R; R$ is a transitive relation, if for any $x, y, z \in U, (x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R; R$ is a serial relation, if for any $x \in U$ there exists $y \in U$ such that $(x, y) \in R; R$ is a reverse serial relation, if for any $y \in U$ there exists $x \in U$ such that $(x, y) \in R$.

Definition 2.2 A relation R is said to be a fuzzy relation, if it is a mapping from $U \times V$ into the unit interval [0, 1], i.e.,

$$R : U \times V \mapsto [0, 1]$$
$$(x, y) \mapsto R(x, y),$$

where $R(x, y)$ is the degree of relation between x and y.

If $U = V$, R is referred to a binary fuzzy relation on $U \times U$.

Let R is a binary fuzzy relation on $U \times U$, R is a reflexive fuzzy relation, if for any $x \in U$, we have $R(x,x) = 1$; R is a symmetric fuzzy relation, if for any $x, y \in U \Rightarrow R(x, y) = R(y, x); R$ is a transitive fuzzy relation, if for any $x, z \in U \Rightarrow R(x, z) \geq \bigvee_{y \in U}(R(y, z) \wedge R(x, z)); R$ is a serial fuzzy relation, if for any $x \in U$ there exists $y \in U$ such that $R(x,y) = 1$; R is a reverse serial fuzzy relation, if for any $y \in U$, there exists $x \in U$ such that $R(x,y) = 1$.

The α cut set and strong α cut set of R will be denoted by R_α and R_{α_+} as follows:

$$R_\alpha = \{(x, y)|R(x, y) \geq \alpha\}; \quad R_{\alpha_+} = \{(x, y)|R(x, y) > \alpha\},$$

where $\alpha \in [0, 1]$.

It is not difficult to find out that for any $\alpha \in [0, 1], R$ is a reflexive (symmetric, transitive, serial and reverse serial, respectively) fuzzy relation, if and only if $R_\alpha(R_{\alpha_+})$ is a reflexive (symmetric, transitive, serial and reverse serial, respectively) relation.

2.2 Fuzzy rough set models over a universe

We will introduce some basic knowledge and notions of the fuzzy rough set theory [39].

The notion of approximation space provides a convenient tool for the rough set theory research. A generalized approximation space is an ordered triple (U, V, R), where U, V are two finite and nonempty sets called two universes, and R is an arbitrary binary relation on $U \times V$. Especially, if $U = V$ and R is an equivalence relation on U, i.e., R is reflexive, symmetric, and transitive, then the approximation space (U, R) is said to be the Pawlak approximation space.

Definition 2.3 Let (U, R) be a Pawlak approximation space, i.e., R is an equivalent relation on U, $[x]_R$ represents the class which including x. For any $A \in F(U)$, denote

$\underline{R}(A)(x) = \min\{A(y)|y \in [x]_R\}$,
$\overline{R}(A)(x) = \max\{A(y)|y \in [x]_R\}$,

where $\underline{R}(A)$ and $\overline{R}(A)$ are called the lower and upper approximation of A in Pawlak approximation space (U, R), respectively.

Proposition 2.1 *Let* (U, R) *be a Pawlak approximation space, then for any* $A, B \in F(U)$, *the lower and upper approximations satisfy the following properties.*

(1) $\underline{R}(A) \subseteq A \subseteq \overline{R}(A)$;
(2) $\underline{R}(A) = \sim R(\sim A), \quad \overline{R}(A) = \sim \underline{R}(\sim A)$;
(3) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B), \quad \overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$;
(4) $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B), \quad \overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$;
(5) $\overline{R}(\underline{R}(A)) = \underline{R}(\underline{R}(A)) = \underline{R}(A)$;
(6) $\underline{R}(\overline{R}(A)) = \overline{R}(\overline{R}(A)) = \overline{R}(A)$.

An approximation space is called a fuzzy information system, if R is a fuzzy reflexive relation on $U \times U$. The (U, R) is called a fuzzy equivalent relation information system, if and only if R is a fuzzy equivalent relation.

Definition 2.4 Let (U, R) be a fuzzy information system, for any $x \in U$,

$$[x] : U \mapsto [0, 1]$$
$$y \mapsto R(x, y).$$

$[x]$ is called the fuzzy neighborhood of x. For any $X \subseteq U$, the fuzzy lower and fuzzy upper approximation of X could be defined as follows:

$$\underline{R}(X)(y) = \min_{x \notin X}(1 - R(x, y)), \quad \overline{R}(X)(y) = \max_{x \in X} R(x, y),$$

where $\underline{R} : P(U) \mapsto F(U)$ and $\overline{R} : P(U) \mapsto F(U)$ are called fuzzy lower and fuzzy upper approximation operators.

From the definition, we can easily find out the following properties.

Proposition 2.2 *Let* (U, R) *be a fuzzy information system and* $X, Y \subseteq U$, *then the fuzzy lower and fuzzy upper approximations satisfy the following properties.*

(1) $\underline{R}(U) = U, \quad \overline{R}(\emptyset) = \emptyset$;
(2) $\underline{R}(X) = \sim \overline{R}(\sim X), \quad \overline{R}(X) = \sim \underline{R}(\sim X)$;
(3) $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y), \quad \overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$;
(4) $\underline{R}(X \cup Y) \supseteq \underline{R}(X) \cup \underline{R}(Y), \quad \overline{R}(X \cap Y) \subseteq \overline{R}(X) \cap \overline{R}(Y)$;
(5) $\underline{R}(X) \subseteq X \subseteq \overline{R}(X)$.

Definition 2.5 Let (U, R) be a fuzzy information system, for any $A \in F(U)$, denote

$$\underline{R}(A)(x) = \wedge\{A(y) \vee (1 - R(x, y)|y \in U)\}, \quad x \in U;$$
$$\overline{R}(A)(x) = \vee A(y) \wedge R(x, y)|y \in U, \quad x \in U,$$

where $\underline{R}(X)$ and $\overline{R}(X)$ are called the fuzzy lower and fuzzy upper approximation of A in (U, R), respectively.

Proposition 2.3 *Let* (U, R) *be a fuzzy information system, then the lower and upper approximations satisfy the following properties.*

(1) $\underline{R}(A) \subseteq A \subseteq \overline{R}(A)$;
(2) $\underline{R}(A) = \sim \overline{R}(\sim A), \quad \overline{R}(A) = \sim \underline{R}(\sim A)$;
(3) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B), \quad \overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$;
(4) $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B), \quad \overline{R}\ (A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$.

2.3 Rough set over two universes

Let (U, V, R) be a generalized approximation space, we can define two mappings $R_s : U \mapsto P(V)$ and $R_p : V \mapsto\ P(U)$

$$R_s(x) = \{y \in V|xRy, x \in U\},$$
$$R_p(y) = \{x \in U|xRy, y \in V\};$$

where $R_s(x)$, $R_p(y)$ denotes all R-related elements to x in V and all R-related elements to y in U, respectively. $R_s(x)$ is called the successor neighborhood of x with respect to R in V and $R_p(y)$ is called the predecessor neighborhood of y with respect to R in U.

Definition 2.6 For any subset X of U and Y of V, the lower and upper approximation of X and Y could be defined as follows:

$$\underline{R}_U(X) = \{y \in V|R_p(y) \subseteq X\},$$
$$\overline{R}_U(X) = \{y \in V|R_p(y) \cap X \neq \emptyset\};$$
$$\underline{R}_V(Y) = \{x \in U|R_s(x) \subseteq Y\},$$
$$\overline{R}_V(Y) = \{x \in U|R_s(x) \cap Y \neq \emptyset\}.$$

If $U = V$ and R is an equivalence relation on U, i.e., (U, R) is a Pawlak approximation space, then

$$R_s(x) = \{y \in U|xRy, x \in U\} = R_p(y)$$
$$= \{x \in U|xRy, y \in U\} = [x]_R.$$

For any subset X of U, the lower and upper approximation of X could be defined as follows:

$$\underline{R}(X) = \cup\{Y \in U/R|Y \subseteq X\} = \{x \in U|[x]_R \subseteq X\},$$
$$\overline{R}(X) = \cup\{Y \in U/R|Y \cap X \neq \emptyset\} = \{x \in U|[x]_R \cap X \neq \emptyset\},$$

where $[x]_R = \{y \in U|R_s(y) = R_s(x)\}$ and $U/R = \{[x]_R|x \in U\}$.

Remark 2.1 It can easily find out that the approximation operators over two universes can be degenerated into Pawlak approximation operators when the two universes satisfy $U = V$ and the relation R is an equivalent relation on U.

3 Fuzzy rough set models over two universes

In this section, we will introduce three types of the rough set models over the different universes.

3.1 Rough fuzzy set in a generalized approximation space

Definition 3.1 Let (*U*, *V*, *R*) be a generalized approximation space, for any $A \in F(U), B \in F(V)$, denote

$$\underline{R}_U(A)(y) = \min\{A(x)|x \in R_p(y)\}, \quad \overline{R}_U(A)(y) = \max\{A(x)|x \in R_p(y)\}, y \in V;$$
$$\underline{R}_V(B)(x) = \min\{B(y)|y \in R_s(x)\}, \quad \overline{R}_V(B)(x) = \max\{B(y)|y \in R_s(x)\}, x \in U,$$

then $\underline{R}_U(A)$ and $\overline{R}_U(A)$ are called the lower and upper approximations of fuzzy set *A* in $F(U), \underline{R}_V(B)$ and $\overline{R}_V(B)$ are called the lower and upper approximations of fuzzy set *B* in *F*(*V*).

If for any $y \in V$ (respectively, $x \in U$), $\underline{R}_U(A)(y) = \overline{R}_U(A)(y)$ (respectively, $\underline{R}_V(B)(x) = \overline{R}_V(B)(x)$), then the fuzzy set *A* (respectively, *B*) is a fuzzy definable set about the generalized approximation space (*U*, *V*, *R*). Otherwise the fuzzy set *A* (respectively, *B*) is a rough set about the generalized approximation space, and *A* (respectively, *B*) is called a rough fuzzy set.

In the following, we employ an example to illustrate the above concepts.

Example 3.1 The relationships of the students and classes are given in Table 1 about some college, $U = \{x_1, x_2, \ldots, x_{10}\}$ is a universe which consists of ten students in some college, *MAC1* (*Mathematic Class 1*), *ENC1* (*English Class 1*), *CHC1* (*Chinese Class 1*), *PHC1* (*Physical Class 1*) are classes of the college. *A* = (0.8, 0.9, 0.7, 0.3, 0.4, 0.6, 0.8, 0.9, 0.2, 0.7) is the excellent degree of these students by the expert, and *B* = {0.9, 0.7, 0.5, 0.3} is the excellent degree of these classes of the college.

We can raise some questions as following:

Question 1 What is the degree of these classes must be excellent according to the excellent degree of these students?

Question 2 What is the degree of these classes may be excellent according to the excellent degree of these students?

Question 3 What is the degree of these students must be excellent according to the excellent degree of these classes?

Question 4 What is the degree of these students may be excellent according to the excellent degree of these classes?

Table 1 The relationships of the students and classes of some college

Class	MAC1	ENC1	CHC1	PHC1
x_1	1	1	0	1
x_2	0	1	1	0
x_3	1	1	1	1
x_4	0	0	1	0
x_5	0	0	1	0
x_6	0	1	1	0
x_7	1	0	1	1
x_8	1	1	0	1
x_9	1	0	0	1
x_{10}	1	1	0	1

Now, we can solve the above questions according to Definition 3.1 as follows:

$$\underline{R}(A) = \{0.2, 0.6, 0.3, 0.2\};$$
$$\overline{R}(A) = \{0.9, 0.9, 0.9, 0.9\};$$
$$\underline{R}(B) = \{0.3, 0.5, 0.3, 0.5, 0.5, 0.5, 0.3, 0.3, 0.3, 0.3\};$$
$$\overline{R}(B) = \{0.9, 0.7, 0.9, 0.5, 0.5, 0.7, 0.9, 0.9, 0.9, 0.9\}.$$

Remark 3.1 In a generalized approximation space, we can find out that the lower and upper approximations of fuzzy set $A \in F(U)$ belong to *F*(*V*), and the lower and upper approximations of fuzzy set $B \in F(V)$ belong to *F*(*U*) according the results of Example 3.1. This property is different from the lower and upper approximations over a universe. What's more, we can obtain the other properties as following.

Theorem 3.1 *Let* (*U*, *V*, *R*) *be a generalized approximation space, for any* $A, A' \in F(U), B, B' \in F(V)$, *we have the following properties*.

(1) $\underline{R}_U(A) = \sim\overline{R}_U(\sim A), \quad \overline{R}_U(A) = \sim\underline{R}_U(\sim A);$
$\underline{R}_V(B) = \sim\overline{R}_V(\sim B), \quad \overline{R}_V(B) = \sim\underline{R}_V(\sim B);$

(2) $\underline{R}_U(A \cap A') = \underline{R}_U(A) \cap \underline{R}_U(A'),$
$\overline{R}_U(A \cup A') = \overline{R}_U(A) \cup \overline{R}_U(A');$
$\underline{R}_V(B \cap B') = \underline{R}_V(B) \cap \underline{R}_V(B'),$
$\overline{R}_V(B \cup B') = \overline{R}_V(B) \cup \overline{R}_V(B');$

(3) $A' \subseteq A \Rightarrow \underline{R}_U(A') \subseteq \underline{R}_U(A), \quad \overline{R}_U(A') \subseteq \overline{R}_U(A);$
$B' \subseteq B \Rightarrow \underline{R}_V(B') \subseteq \underline{R}_V(B), \overline{R}_V(B') \subseteq \overline{R}_V(B);$

(4) $\underline{R}(A \cup A') \supseteq \underline{R}(A) \cup \underline{R}(A'), \ \overline{R}(A \cap A') \subseteq \overline{R}(A) \cap \overline{R}(A');$
$\underline{R}(B \cup B') \supseteq \underline{R}(B) \cup \underline{R}(B'), \ \overline{R}(B \cap B') \subseteq \overline{R}(B) \cap \overline{R}(B');$

(5) $\underline{R}_U(A) \subseteq \overline{R}_U(A), \quad \underline{R}_V(B) \subseteq \overline{R}_V(B).$

Proof We only need to prove the first part of each property as the similarity of the above properties.

(1) $\forall y \in V$, according to Definition 3.1, we can obtain

$$\begin{aligned}\underline{R}_U(\sim A)(y) &= \min\{\sim A(x)|x \in R_p(y)\}\\ &= \min\{1 - A(x)|x \in R_p(y)\}\\ &= 1 - \max\{A(x)|x \in R_p(y)\}\\ &= \sim \overline{R}_U A(y).\end{aligned}$$

So we can have $\overline{R}_U(A) = \sim \underline{R}_U(\sim A)$.
The property $\underline{R}_U(A) = \sim \overline{R}_U(\sim A)$ can be proved similarly.

(2) $\forall y \in V$, we can have

$$\begin{aligned}\underline{R}_U(A \cap A')(y) &= \min\{A(x) \wedge A'(x)|x \in R_p(y)\}\\ &= \min\{A(x)|x \in R_p(y)\}\\ &\quad \wedge \min\{A'(x)|x \in R_p(y)\}\\ &= \underline{R}_U(A)(y) \cap \underline{R}_U(A')(y).\end{aligned}$$

Hence, we can obtain $\underline{R}_U(A \cap A') = \underline{R}_U(A) \cap \underline{R}_U(A')$.

(3) According to the definitions of fuzzy lower and fuzzy upper approximation, (3) holds.

(4) It is easy to prove by the property (3).

(5) $\forall y \in \underline{R}_U(A)$, we can have

$$\begin{aligned}\underline{R}_U(A)(y) &= \min\{A(x)|y \in R_p(y)\} \le \overline{R}_U(A)(y)\\ &= \max\{A(x)|y \in R_p(y)\}.\end{aligned}$$

Therefore, $\underline{R}_U(A) \subseteq \overline{R}_U(A)$.

Definition 3.2 Let (*U*, *V*, *R*) be a generalized approximation space, for any $A \in F(U), B \in F(V)$, denote

$$\underline{R}_U(A_\alpha) = \{y|R_p(y) \subseteq A_\alpha\}, \quad \overline{R}_U(A_\alpha) = \{y|R_p(y) \cap A_\alpha \neq \emptyset\};$$
$$\underline{R}_V(B_\alpha) = \{x|R_s(x) \subseteq B_\alpha\}, \quad \overline{R}_V(B_\alpha) = \{x|R_s(x) \cap B_\alpha \neq \emptyset\},$$

where $\alpha \in [0, 1], \underline{R}_U(A_\alpha)$ and $\overline{R}_U(A_\alpha)$ are called the lower and upper approximation of A_α on the universe $U, \underline{R}_V(B_\alpha)$ and $\overline{R}_V(B_\alpha)$ are called the lower and upper approximation of B_α on the universe *V*.

Theorem 3.2 *Let* (*U*, *V*, *R*) *be a generalized approximation space, if* $\alpha < \beta$, *we can obtain*

$$\underline{R}_U(A_\beta) \subseteq \underline{R}_U(A_\alpha), \quad \overline{R}_U(A_\beta) \subseteq \overline{R}_U(A_\alpha);$$
$$\underline{R}_V(B_\beta) \subseteq \underline{R}_V(B_\alpha), \quad \overline{R}_V(B_\beta) \subseteq \overline{R}_V(B_\alpha).$$

Proof Since $\alpha < \beta$, so $A_\beta \subseteq A_\alpha$. For any $y \in \underline{R}_U(A_\beta)$, we can have $R_p(y) \subseteq A_\beta$. Thus, $R_p(y) \subseteq A_\alpha \Leftrightarrow y \in \underline{R}_U(A_\alpha)$. I.e., $\underline{R}_U(A_\beta) \subseteq \underline{R}_U(A_\alpha)$.

The property $\overline{R}_U(A_\beta) \subseteq \overline{R}_U(A_\alpha), \underline{R}_V(B_\beta) \subseteq \underline{R}_V(B_\alpha)$ and $\overline{R}_V(B_\beta) \subseteq \overline{R}_V(B_\alpha)$ can be proved similarly.

According to the Definition 3.2, we can define two pairs of fuzzy sets as follows:

$$\underline{R}'_U(A)(y) = \vee\{\alpha|y \in \underline{R}_U(A_\alpha)\} = \vee\{\alpha|R_p(y) \subseteq A_\alpha\},$$
$$\overline{R}'_U(A)(y) = \vee\{\alpha|y \in \overline{R}_U(A_\alpha)\} = \vee\{\alpha|R_p(y) \cap A_\alpha \neq \emptyset\},$$
$$\underline{R}'_V(B)(x) = \vee\{\alpha|x \in \underline{R}_V(B_\alpha)\} = \vee\{\alpha|R_s(x) \subseteq B_\alpha\},$$
$$\overline{R}'_V(B)(x) = \vee\{\alpha|x \in \overline{R}_V(B_\alpha)\} = \vee\{\alpha|R_s(x) \cap B_\alpha \neq \emptyset\}.$$

Then we can obtain the properties in the following.

Theorem 3.3 *Let* (*U*, *V*, *R*) *be a generalized approximation space, for any* $A \in F(U), B \in F(V)$, *then*

$$\underline{R}_U(A) = \underline{R}'_U(A), \quad \overline{R}_U(A) = \overline{R}'_U(A);$$
$$\underline{R}_V(B) = \underline{R}'_V(B), \quad \overline{R}_V(B) = \overline{R}'_V(B).$$

Proof For any $y \in V$, denote

$$\alpha_1 = \underline{R}_U(A)(y) = \min\{A(x)|x \in R_p(y)\},$$
$$\alpha_2 = \underline{R}'_U(A)(y) = \max\{\alpha|R_p(y) \subseteq A_\alpha\}.$$

Let α satisfy $R_p(y) \subseteq A_\alpha$, if $x \in R_p(y)$, then $A(x) \ge \alpha$ and $\min\limits_{x \in R_p(y)} A(x) \ge \alpha$. So $\alpha_1 \ge \alpha$, therefore $\alpha_1 \ge \alpha_2$.

On the other hand, for any $\alpha > \alpha_2$, according to the definition of α_2, we can know that there exists $x \in R_p(y)$, s.t. $x \notin A_\alpha$, i.e., $\alpha_1 \le A(x) < \alpha$, thus $\alpha > \alpha_1$, by the arbitrary of $\alpha > \alpha_2$, we can obtain $\alpha_2 \ge \alpha_1$. Hence $\underline{R}_U(A) = \underline{R}'_U(A)$.

The properties $\overline{R}_U(A) = \overline{R}'_U(A), \underline{R}_V(B) = \underline{R}'_V(B)$ and $\overline{R}_V(B) = \overline{R}'_V(B)$ can be proved similarly.

Remark 3.2 We can obtain the same consequences for the strong cut of *A*.

Definition 3.3 Let (*U*, *V*, *R*) be a generalized approximation space, $A_1, A_2 \in F(U), B_1, B_2 \in F(V)$.

If $\underline{R}_U(A_1) = \underline{R}_U(A_2)$, then A_1 and A_2 are called lower rough equivalences of *U*, denoted by $A_1 \eqsim_U A_2$,

If $\overline{R}_U(A_1) = \overline{R}_U(A_2)$, then A_1 and A_2 are called upper rough equivalences of *U*, denoted by $A_1 \simeq_U A_2$,

If $\underline{R}_U(A_1) = \underline{R}_U(A_2)$ and $\overline{R}_U(A_1) = \overline{R}_U(A_2)$, then A_1 and A_2 are called rough equivalences of *U*, denoted by $A_1 \approx_U A_2$,

If $\underline{R}_V(B_1) = \underline{R}_V(B_2)$, then B_1 and B_2 are called lower rough equivalences of *V*, denoted by $B_1 \eqsim_V B_2$,

If $\overline{R}_V(B_1) = \overline{R}_V(B_2)$, then B_1 and B_2 are called upper rough equivalences of *V*, denoted by $B_1 \simeq_V B_2$,

If $\underline{R}_V(B_1) = \underline{R}_V(B_2)$ and $\overline{R}_V(B_1) = \overline{R}_V(B_2)$, then B_1 and B_2 are called rough equivalences of *V*, denoted by $B_1 \approx_V B_2$.

Proposition 3.1 *Let* (*U*, *V*, *R*) *be a generalized approximation space,* $A_1, A_2, A'_1, A'_2 \in F(U)$; $B_1, B_2, B'_1, B'_2 \in F(V)$, *then*

(1) $A_1 \eqsim_U A_2 \Leftrightarrow (A_1 \cap A_2) \eqsim_U A_2, (A_1 \cap A_2) \eqsim UA_1$;
$B_1 \eqsim_V B_2 \Leftrightarrow (B_1 \cap B_2) \eqsim_V B_2, (B_1 \cap B_2) \eqsim_V B_1$;

(2) $A_1 \simeq_U A_2 \Leftrightarrow (A_1 \cup A_2) \simeq_U A_2, (A_1 \cup A_2) \simeq_U A_1$;
$B_1 \simeq_V B_2 \Leftrightarrow (B_1 \cup B_2) \simeq_V B_2, (B_1 \cup B_2) \simeq_V B_1$;
(3) *If* $A_1 \eqsim_U A'_1, A_2 \eqsim_U A'_2$, *then* $(A_1 \cap A_2) \eqsim_U A'_1 \cap A'_2$;
if $A_1 \simeq_U A'_1, A_2 \simeq_U A'_2$, *then* $(A_1 \cup A_2) \simeq_U A'_1 \cup A'_2$;
if $B_1 \eqsim_V B'_1, B_2 \eqsim_V B'_2$, *then* $(B_1 \cap B_2) \eqsim_V B'_1 \cap B'_2$;
if $B_1 \simeq_V B'_1, B_2 \simeq_V B'_2$, *then* $(B_1 \cup B_2) \simeq_V B'_1 \cup B'_2$.
(4) if $A_1 \eqsim_U \emptyset$ *or* $A'_1 \eqsim_U \emptyset$, *then* $A_1 \cap A'_1 \eqsim_U \emptyset$;
if $B_1 \eqsim_V \emptyset, B'_1 \eqsim_V \emptyset$, *then* $B_1 \cap B'_1 \eqsim_V \emptyset$.
(5) *If* $A_1 \simeq_U U$ *or* $A'_1 \simeq_U U$, *then* $A_1 \cup A'_1 \simeq_U U$;
if $B_1 \simeq_V V or B'_1 \simeq_V V$, *then* $B_1 \cup B'_1 \simeq_V V$.
(6) If $A_1 \subseteq A'_1$ *and* $A'_1 \eqsim_U \emptyset$, *then* $A_1 \eqsim_U \emptyset$;
if $B_1 \subseteq B'_1$ *and* $B'_1 \eqsim_V \emptyset$, *then* $B_1 \eqsim_V \emptyset$.
(7) *If* $A_1 \subseteq A'_1$ *and* $A_1 \simeq_U U$, *then* $A'_1 \simeq_U U$;
if $B_1 \subseteq B'_1$ *and* $B_1 \simeq_V V$, *then* $B'_1 \simeq_V V$.

Proof Straightforward.

Theorem 3.4 *Let* (U, V, R) *be a generalized approximation space*, $A \in F(U)$, $B \in F(V)$, *then*

(1) $\underline{R}_U(A) = \{\cap A' \in F(U) | A \eqsim_U A'\}$,
$\underline{R}_V(B) = \{\cap B' \in F(V) | B \eqsim_V B'\}$;
(2) $\overline{R}_U(A) = \{\cup A' \in F(U) | A \simeq_U A'\}$,
$\overline{R}_V(B) = \{\cup B' \in F(V) | B \simeq_V B'\}$.

Proof We can obtain them according to Proposition 3.1.

Theorem 3.5 *Let* (U, V, R) *be a generalized approximation space*, $A \in F(U)$, *for any* $0 \leq \alpha, \beta \leq 1$, *if* R *is a reverse serial relation on* $U \times V$, *denote*

$\underline{R}_U(A))_\alpha = \{y \in V | (\underline{R}_U(A))(y) \geq \alpha\}$,
$(\overline{R}_U(A))_\beta = \{y \in V | (\overline{R}_U(A))(y) \geq \beta\}$;

then

(1) $(\underline{R}_U(A))_\alpha \supseteq \underline{R}_U((A)_\alpha)$, $\quad (\overline{R}_U(A))_\beta \supseteq \overline{R}_U((A)_\beta)$.
(2) $\alpha \geq \beta \Rightarrow (\underline{R}_U(A))_\alpha \subseteq (\overline{R}_U(A))_\alpha \subseteq (\overline{R}_U(A))_\beta$.

Proof

(1) For any $y \in \underline{R}_U(A_\alpha) \Rightarrow \emptyset \neq R_p(y) \subseteq A_\alpha \Rightarrow \forall x \in R_p(y) \subseteq A_\alpha \Rightarrow \forall x \in R_p(y), A(x) \geq \alpha \Rightarrow \min A(x) | x \in R_p(y) \geq \alpha \Rightarrow \underline{R}_U(A)(y) \geq \alpha \Rightarrow y \in (\underline{R}_U(A))_\alpha$. Thus, we can have $(\underline{R}_U(A))_\alpha \supseteq \underline{R}_U((A)_\alpha)$.
The property $(\underline{R}_U(A))_\alpha \supseteq \underline{R}_U((A)_\alpha)$. can be proved similarly.
(2) Since $\alpha \geq \beta$, we can obtain $\forall y \in (\underline{R}_U(A))_\alpha \Rightarrow \min\{A(x) | x \in R_p(y)\} \geq \alpha \Rightarrow \max\{A(x) | x \in R_p(y)\} \geq \alpha \Rightarrow y \in (\overline{R}_U(A))_\alpha \Rightarrow \max\{A(x) | x \in R_p(y)\} \geq \alpha \geq \beta \Rightarrow y \in (\overline{R}_U(A))_\beta$, therefore, $(\underline{R}_U(A))_\alpha \subseteq (\overline{R}_U(A))_\alpha \subseteq (\overline{R}_U(A))_\beta$.

Theorem 3.6 *Let* (U, V, R) *be a generalized approximation space*, $B \in F(V)$, *for any* $0 \leq \alpha, \beta \leq 1$, *if* R *is a serial relation on* $U \times V$, *denote*

$\underline{R}_V(B))_\alpha = \{x \in U | (\underline{R}_V(B))(x) \geq \alpha\}$,
$(\overline{R}_V(B))_\beta = \{x \in U | (\overline{R}_V(B))(x) \geq \beta\}$,

then

(1) $(\underline{R}_V(B))_\alpha \supseteq \underline{R}_V(B_\alpha)$, $\quad (\overline{R}_V(B))_\beta \supseteq \overline{R}_V(B_\beta)$.
(2) $\alpha \geq \beta \Rightarrow (\underline{R}_U(A))_\alpha \subseteq (\overline{R}_U(A))_\beta$,
$(\underline{R}_V(B))_\alpha \subseteq (\overline{R}_V(B))_\beta$.

Proof The proof is similar to Theorem 3.5.

3.2 Rough set in a fuzzy approximation space

Let U and V be two non-empty finite sets called double universes of discourse. R be an arbitrary fuzzy relation on $U \times V$, the ordered triple (U, V, R) is called fuzzy approximation space.

Definition 3.4 Let (U, V, R) be a fuzzy approximation space, for $X \subseteq U$ $Y \subseteq V$, denote

$$\underline{R}_U(X)(y) = \min_{x \notin X}(1 - R(x,y)), \overline{R}_U(X)(y) = \max_{x \in X} R(x,y) y \in V;$$

$$\underline{R}_V(Y)(x) = \min_{y \notin Y}(1 - R(x,y)), \overline{R}_V(Y)(x) = \max_{y \in Y} R(x,y)\ x \in U;$$

then $\underline{R}_U(X)$ is called the fuzzy lower approximation of the set X on the universe U, $\overline{R}_U(X)$ is called the fuzzy upper approximation of the set X on the universe U, $\underline{R}_V(Y)$ is called the fuzzy lower approximation of the set Y on the universe V, and $\overline{R}_V(Y)$ is called the fuzzy upper approximation of the set Y on the universe V.

If for any $y \in V$(respectively, $x \in U$), $\underline{R}_U(X)(y) = \overline{R}_U(X)(y)$ (respectively, $\underline{R}_V(Y)(x) = \overline{R}_V(Y)(x)$), then the set X (respectively, Y) is definable with respect to fuzzy approximation space (U, V, R). Otherwise the set X (respectively, Y) is rough set with respect to the fuzzy approximation space.

Example 3.2 The student comprehensive evaluation system of some college is given in Table 2, $U = \{Mental\ quality, Intelligent\ quality, Physical\ quality\}$ is a universe, $V = \{Best, Better, Good, Bad\}$ is the evaluation set as the other universe, and the relationships of U and V are follows:

Table 2 The student comprehensive evaluation system of some college

(U, V)	Best	Better	Good	Bad
Mental quality	0.7	0.6	0.5	0.8
Intelligent quality	0.6	0.9	1	0.8
Physical quality	0.7	0.4	0.3	0.2

Let's consider the lower and upper approximations of the set $X = \{Mental\quad quality, Intelligent\ quality\}$ and $Y = \{Best, Better, Good\}$.

Obviously, we can obtain

$$\underline{R}_U(X) = (0.3, 0.6, 0.7, 0.8); \quad \overline{R}_U(X) = (0.7, 0.9, 1, 0.8);$$
$$\underline{R}_V(Y) = (0.2, 0.2, 0.8); \quad \overline{R}_V(Y) = (0.7, 1, 0.7).$$

Remark 3.3 In a fuzzy approximation space, we can find out that the lower and upper approximations of fuzzy set $X \in U$ belong to $F(V)$, and the lower and upper approximations of fuzzy set $Y \in V$ belong to $F(U)$ according the results of Example 3.2. This property is different from the lower and upper approximations over a universe. What's more, the property $\underline{R}_V(Y) \subseteq \overline{R}_V(Y)$ is not true in fuzzy approximation space.

Based on the above definitions, some properties of lower and upper approximation operators will be obtained.

Theorem 3.7 *Let (U, V, R) be a fuzzy approximation space, for any $X \subseteq U, X' \subseteq U, Y \subseteq V, Y' \subseteq V$, we have the following properties.*

(1) $\underline{R}_U(U) = V$, $\overline{R}_U(\emptyset) = \emptyset$; $\underline{R}_V(V) = U$, $\overline{R}_V(\emptyset) = \emptyset$;
(2) $\underline{R}_U(X) = \sim\overline{R}_U(\sim X), \overline{R}_U(X) = \sim\underline{R}_U(\sim X)$;
$\underline{R}_V(Y) = \sim\overline{R}_V(\sim Y), \overline{R}_V(Y) = \sim\underline{R}_V(\sim Y)$;
(3) $\underline{R}_U(X\cap X') = \underline{R}_U(X)\cap\underline{R}_U(X')$;
$\overline{R}_U(X\cup X') = \overline{R}_U(X)\cup\overline{R}_U(X')$;
$\underline{R}_V(Y\cap Y') = \underline{R}_V(Y)\cap\underline{R}_V(Y')$;
$\overline{R}_V(Y\cup Y') = \overline{R}_V(Y)\cup\overline{R}_V(Y')$;
(4) $X' \subseteq X \Rightarrow \underline{R}_U(X') \subseteq \underline{R}_U(X), \overline{R}_U(X') \subseteq \overline{R}_U(X)$;
$Y' \subseteq Y \Rightarrow \underline{R}_V(Y') \subseteq \underline{R}_V(Y), \overline{R}_V(Y') \subseteq \overline{R}_V(Y')$;
(5) $\underline{R}_U(X\cup X') \supseteq \underline{R}_U(X)\cup\underline{R}_U(X')$,
$\overline{R}_U(X\cap X') \subseteq \overline{R}_U(X)\cap\overline{R}(X')$;
$\underline{R}_V(Y\cup Y') \supseteq \underline{R}_V(Y)\cup\underline{R}_V(Y')$,
$\overline{R}_V(Y\cap Y') \subseteq \overline{R}_V(Y)\cap\overline{R}_V(Y')$.

Proof According to the similarity of these characteristics, we only need to prove the first portion of each properties.

(1) According to the definitions of fuzzy lower and fuzzy upper approximation, we can obtain, $\forall y \in V$,

$$\underline{R}_U(U)(y) = \min_{x\notin U}(1 - R(x,y)) = 1,$$
$$\overline{R}_U(\emptyset)(y) = \max_{x\in\emptyset} R(x,y) = 0.$$

Therefore, $\forall y \in V$, we know $y \in \underline{R}_U(U), y \in \sim\underline{R}_U(\emptyset)$. Hence, we can obtain $\underline{R}_U(U) = V, \overline{R}_U(\emptyset) = \emptyset$.

(2) $\forall\ y \in V$, according to the definitions of fuzzy lower and fuzzy upper approximation, we can obtain

$$\underline{R}_U(\sim X)(y) = \min_{x\in X}(1 - R(x,y)) = 1 - \max_{x\in X} R(x,y) = \sim\overline{R}_U(X)(y).$$

So we can have $\overline{R}_U(X) = \sim\underline{R}_U(\sim X)$.

The property $\underline{R}_U(X) = \sim\overline{R}_U(\sim X)$ can be proved similarly.

(3) For $\forall y \in V$, we can have

$$\underline{R}_U(X\cap X')(y) = \min_{x\notin(X\cap X')}(1 - R(x,y)) = (\min_{x\notin X}(1 - R(x,y))) \wedge (\min_{x\notin X'}(1 - R(x,y))) = (\underline{R}_U(X)\cap\underline{R}_U(X'))(y).$$

Hence, we can obtain $\underline{R}_U(X\cap X') = \underline{R}_U(X)\cap\underline{R}_U(X')$.

(4) According to the definitions of fuzzy lower and fuzzy upper approximation, (4) holds.
(5) According to the property (4), obviously, (5) holds.

Theorem 3.8 *Let (U, V, R) be a fuzzy approximation space, $\forall X \subseteq U$, $\forall Y \subseteq V$, we have*

$$\underline{R}_U(X) = \overline{R}_U(X) \Leftrightarrow \forall y \in V, \max_{x\notin X} R(x,y) + \max_{x\in X} R(x,y) = 1;$$
$$\underline{R}_V(Y) = \overline{R}_V(Y) \Leftrightarrow \forall x \in U, \max_{y\notin Y} R(x,y) + \max_{y\in Y} R(x,y) = 1.$$

Proof From the definitions of fuzzy lower and upper approximation, $\forall y \in V$, we have

$$\begin{aligned}\underline{R}_U(X) = \overline{R}_U(X) &\Leftrightarrow \underline{R}_U(X)(y) = \overline{R}_U(X)(y)\\ &\Leftrightarrow \min_{x\notin X}(1 - R(x,y)) = \max_{x\in X} R(x,y)\\ &\Leftrightarrow 1 - \max_{x\notin X} R(x,y) = \max_{x\in X} R(x,y)\\ &\Leftrightarrow \max_{x\notin X} R(x,y) + \max_{x\in X} R(x,y) = 1.\end{aligned}$$

The theorem has been proved completely.

Definition 3.5 Let (U, V, R) be a fuzzy approximation space, $\forall x \in U$, $\forall y \in V$,

$$[x] : V \mapsto [0,1] \quad [y] : U \mapsto [0,1]$$
$$y \rightarrow R(x,y) \quad x \rightarrow R(x,y).$$

$[x]$ is called the fuzzy neighborhood of x on the universe U and $[y]$ is called the fuzzy neighborhood of y on the universe V.

Theorem 3.9 *Let (U, V, R) be a fuzzy approximation space, for any $X \subseteq U$, $Y \subseteq V$, the following expressions hold:*

$$\underline{R}_U(X) = \bigcap_{x\notin X}(\sim[x]), \quad \overline{R}_U(X) = \bigcup_{x\in X}([x]);$$
$$\underline{R}_V(Y) = \bigcap_{y\notin Y}(\sim[y]), \quad \overline{R}_V(Y) = \bigcup_{y\in Y}([y]).$$

Proof It is easy to prove by the definitions of fuzzy neighborhood.

Definition 3.6 Let (U, V, R) be a fuzzy approximation space, $X_1, X_2 \subseteq U$, $Y_1, Y_2 \subseteq V$.

If $\underline{R}_U(X_1) = \underline{R}_U(X_2)$, then X_1 and X_2 are called fuzzy lower rough equivalences of U, denoted by $X_1 \eqsim_U X_2$,

If $\overline{R}_U(X_1) = \overline{R}_U(X_2)$, then X_1 and X_2 are called fuzzy upper rough equivalences of U, denoted by $X_1 \simeq_U X_2$,

If $\underline{R}_U(X_1) = \underline{R}_U(X_2)$ and $\overline{R}_U(X_1) = \overline{R}_U(X_2)$, then X_1 and X_2 are called fuzzy rough equivalences of U, denoted by $X_1 \approx_U X_2$,

If $\underline{R}_V(Y_1) = \underline{R}_V(Y_2)$, then Y_1 and Y_2 are called fuzzy lower rough equivalences of V, denoted by $Y_1 \eqsim_V Y_2$,

If $\overline{R}_V(Y_1) = \overline{R}_V(Y_2)$, then Y_1 and Y_2 are called fuzzy upper rough equivalences of V, denoted by $Y_1 \simeq_V Y_2$,

If $\underline{R}_V(Y_1) = \underline{R}_V(Y_2)$ and $\overline{R}_V(Y_1) = \overline{R}_V(Y_2)$, then Y_1 and Y_2 are called fuzzy rough equivalences of V, denoted by $Y_1 \approx_V Y_2$.

Proposition 3.2 *Let (U, V, R) be a fuzzy approximation space, $X_1, X_2, X_1', X_2' \subseteq U; Y_1, Y_2, Y_1', Y_2' \subseteq V$, then*

(1) $X_1 \eqsim_U X_2 \Leftrightarrow (X_1 \cap X_2) \eqsim_U X_2, (X_1 \cap X_2) \eqsim_U X_1$;
$Y_1 \eqsim_V Y_2 \Leftrightarrow (Y_1 \cap Y_2) \eqsim_V Y_2, (Y_1 \cap Y_2) \eqsim_V Y_1$;

(2) $X_1 \simeq_U X_2 \Leftrightarrow (X_1 \cup X_2) \simeq_U X_2, (X_1 \cup X_2) \simeq_U X_1$;
$Y_1 \simeq_V Y_2 \Leftrightarrow (Y_1 \cup Y_2) \simeq_V Y_2, (Y_1 \cup Y_2) \simeq_V Y_1$;

(3) $X_1 \eqsim_U X_1', X_2 \eqsim_U X_2'$, *then* $(X_1 \cap X_2) \eqsim_U X_1' \cap X_2'$;
$X_1 \simeq_U X_1', X_2 \simeq_U X_2'$, *then* $(X_1 \cup X_2) \simeq_U X_1' \cup X_2'$;
$Y_1 \eqsim_V Y_1', Y_2 \eqsim_V Y_2'$, *then* $(Y_1 \cap Y_2) \eqsim_V Y_1' \cap Y_2'$;
$Y_1 \simeq_V Y_1', Y_2 \simeq_V Y_2'$, *then* $(Y_1 \cup Y_2) \simeq_V Y_1' \cup Y_2'$.

(4) *If* $X_1 \eqsim_U \emptyset$ *or* $X_1' \eqsim_U \emptyset$, *then* $X_1 \cap X_1' \eqsim_U \emptyset$;
if $Y_1 \eqsim_V \emptyset$ *or* $Y_1' \eqsim_V \emptyset$, *then* $Y_1 \cap Y_1' \eqsim_V \emptyset$.

(5) *If* $X_1 \simeq_U U$ *or* $X_1' \simeq_U U$, *then* $X_1 \cup X_1' \simeq_U U$;
if $Y_1 \simeq_V V$ *or* $Y_1' \simeq_V V$, *then* $Y_1 \cup Y_1' \simeq_V V$.

(6) *If* $X_1 \subseteq X_1'$ *and* $X_1' \eqsim_U \emptyset$, *then* $X_1 \eqsim_U \emptyset$;
if $Y_1 \subseteq Y_1'$ *and* $Y_1' \eqsim_V \emptyset$, *then* $Y_1 \eqsim_V \emptyset$.

(7) *If* $X_1 \subseteq X_1'$ *and* $X_1 \simeq_U U$, *then* $X_1' \simeq_U U$;
if $Y_1 \subseteq Y_1'$ *and* $Y_1 \simeq_V V$, *then* $Y_1' \simeq_V V$.

Proof Straightforward.

Definition 3.7 Let (U, V, R) be a fuzzy approximation space, the α cut relation and strong α cut relation of R are denoted as R_a and R_{a_+}, the lower and upper approximation of $X \subseteq U$ and $Y \subseteq V$ based on R_a and R_{a_+} could be defined as follows:

$$\underline{R}_U^{\alpha}(X) = \{y \in V | R_p^{\alpha}(y) \subseteq X\}; \quad \overline{R}_U^{\alpha}(X) = \{y \in V | R_p^{\alpha}(y) \cap X \neq \emptyset\};$$
$$\underline{R}_V^{\alpha}(Y) = \{x \in U | R_s^{\alpha}(x) \subseteq Y\}; \quad \overline{R}_V^{\alpha}(Y) = \{x \in U | R_s^{\alpha}(x) \cap Y \neq \emptyset\};$$
$$\underline{R}_U^{\alpha_+}(X) = \{y \in V | R_p^{\alpha_+}(y) \subseteq X\}; \quad \overline{R}_U^{\alpha_+}(X) = \{y \in V | R_p^{\alpha_+}(y) \cap X \neq \emptyset\};$$
$$\underline{R}_V^{\alpha_+}(Y) = \{x \in U | R_s^{\alpha_+}(x) \subseteq Y\}; \quad \overline{R}_V^{\alpha_+}(Y) = \{x \in U | R_s^{\alpha_+}(x) \cap Y \neq \emptyset\};$$

where

$$R_s^{\alpha}(x) = \{y \in V | R(x,y) \geq \alpha\}; \quad R_s^{\alpha_+}(x) = \{y \in V | R(x,y) > \alpha\};$$
$$R_p^{\alpha}(y) = \{x \in U | R(x,y) \geq \alpha\}; \quad R_p^{\alpha_+}(y) = \{x \in U | R(x,y) > \alpha\}.$$

$\underline{R}_U^{\alpha}(X)$, $\overline{R}_U^{\alpha}(X)$ are called the α level lower and upper approximation of the set X on the universe U, $\underline{R}_V^{\alpha}(Y)$, $\overline{R}_V^{\alpha}(Y)$ are named the α level lower and upper approximation of the set Y on the universe V, $\underline{R}_U^{\alpha_+}(X)$, $\overline{R}_U^{\alpha_+}(X)$ are called the strong α level lower and upper approximation of the set X on the universe U, and $\underline{R}_V^{\alpha_+}(Y)$, $\overline{R}_V^{\alpha_+}(Y)$ are named the strong α level lower and upper approximation of the set Y on the universe V.

Theorem 3.10 *Let (U, V, R) be a fuzzy approximation space, we can have*

$$\overline{R}_U^{\alpha}(X) = (\overline{R}_U(X))_{\alpha}; \overline{R}_V^{\alpha}(Y) = (\overline{R}_V(Y))_{\alpha},$$

where $X \subseteq U$, $Y \subseteq V, \alpha \in [0,1]$.

Proof For $\forall y \in \overline{R}_{\alpha}(X)$, we can have $R_p^{\alpha}(y) \cap X \neq \emptyset$, that is to say $\exists x \in X$ s.t. $R(x,y) \geq \alpha$. So we can obtain $\max_{x \in X} R(x,y) \geq \alpha$, i.e., $y \in (\overline{R}(X))_{\alpha}$.

Conversely, if $y \in (\overline{R}(X))_{\alpha}$, then $\max_{x \in X} R(x,y) \geq \alpha$, that is to say $\exists x \in X$ s.t. $R(x,y) \geq \alpha$, i.e., $R_p^{\alpha}(y) \cap X \neq \emptyset$, Therefore, $y \in \overline{R}_{\alpha}(X)$.

The theorem only holds for the upper approximation, not for the lower approximation. In the following, we will give an example.

Example 3.3 From Example 3.2, if we take $\alpha = 0.3$, then $R_{0.3}$ is a general relation on $U \times V$ in Table 3:

We have got

$$(\underline{R}_U(X))_{0.3} = V, \qquad (\overline{R}_U(X))_{0.3} = V;$$
$$(\underline{R}_V(Y))_{0.3} = \{Physical\ quality\}, \qquad (\overline{R}_V(Y))_{0.3} = U;$$
$$\underline{R}_U^{0.3}(X) = \{Bad\}, \qquad \overline{R}_U^{0.3}(X) = V;$$
$$\underline{R}_V^{0.3}(Y) = \{Intelligent\ quality, Physical\ quality\}, \qquad \overline{R}_V^{0.3}(Y) = U.$$

Hence, the following is obviously true

$$\underline{R}_U^{0.3}(X) \neq (\underline{R}_U(X))_{0.3}, \quad \overline{R}_U^{0.3}(X) = (\overline{R}_U(X))_{0.3};$$
$$\underline{R}_V^{0.3}(Y) \neq (\underline{R}_V(Y))_{0.3}, \quad \overline{R}_V^{0.3}(Y) = (\overline{R}_V(Y))_{0.3}.$$

Theorem 3.11 *Let (U, V, R) be a fuzzy approximation space, we can have*

$$\underline{R}_U^{(1-\alpha)_+}(X) = (\underline{R}_U(X))_{\alpha}; \quad \underline{R}_V^{(1-\alpha)_+}(Y) = (\underline{R}_V(Y))_{\alpha},$$

where $X \subseteq U$, $Y \subseteq V, \alpha \in [0,1]$.

Proof $\forall y \in \underline{R}_U^{(1-\alpha)_+}(X)$, we can have $R_p^{(1-\alpha)_+}(y) \subseteq X$, that is to say $\forall x \in U$, if $R(x, y) > 1 - \alpha$ then $x \in X$. So we can obtain $\min_{x \notin X}(1 - R(x,y)) \geq \alpha$ i.e., $y \in (\underline{R}_U(X))_{\alpha}$.

Table 3 The relation $R_{0.3}$ on $U \times V$

$R_{0.3}$	Best	Better	Good	Bad
Mental quality	1	1	1	1
Intelligent quality	1	1	1	1
Physical quality	1	1	1	0

Conversely, if $y \in (\underline{R}_U(X))_\alpha$, then $\min_{x \notin X}(1 - R(x,y)) \geq \alpha$, that is to say $\forall x \notin X$ s.t. $1 - R(x,y) \geq \alpha$. I.e., $\forall\ x \in U$, if $R(x,y) > 1 - \alpha$ then $x \in X$. Therefore, $y \in \underline{R}_\alpha(X)$.

Theorem 3.12 *Let (U, V, R) be a fuzzy approximation space, if $\alpha < \beta, \forall X \subseteq U, Y \subseteq V$ we can have*

(1) $\underline{R}_U^\alpha(X) \subseteq \underline{R}_U^\beta(X), \quad \overline{R}_U^\beta(X) \subseteq \overline{R}_U^\alpha(X),$
$\underline{R}_U^{\alpha_+}(X) \subseteq \underline{R}_U^{\beta_+}(X), \quad \overline{R}_U^{\beta_+}(X) \subseteq \overline{R}_U^{\alpha_+}(X);$

(2) $\underline{R}_V^\alpha(Y) \subseteq \underline{R}_V^\beta(Y), \quad \overline{R}_V^\beta(Y) \subseteq \overline{R}_V^\alpha(Y),$
$\underline{R}_V^{\alpha_+}(Y) \subseteq \underline{R}_V^{\beta_+}(Y), \quad \overline{R}_V^{\beta_+}(Y) \subseteq \overline{R}_V^{\alpha_+}(Y).$

Proof According to the similarity of the above properties, we only need to prove the first part of each property.

(1) For any $y \in \underline{R}_U^\alpha(X)$, i.e., $R_p^\alpha(y) \subseteq X$. Note that $\alpha < \beta$, thus, $R^\beta \subseteq R^\alpha$ and $R_p^\beta(y) \subseteq R_p^\alpha(y)$, that is to say $y \in \underline{R}_U^\beta(X)$. Therefore, the properties $\underline{R}_U^\alpha(X) \subseteq \underline{R}_U^\beta(X)$ and $\underline{R}_U^{\alpha_+}(X) \subseteq \underline{R}_U^{\beta_+}(X)$ can be proved similarly.

(2) It is similar to prove that $\underline{R}_V^\alpha(Y) \subseteq \underline{R}_V^\beta(Y)$ and $\underline{R}_V^{\alpha_+}(Y) \subseteq \underline{R}_V^{\beta_+}(Y)$.

Definition 3.8 Let (U, V, R) be a fuzzy approximation space, for any $X \subseteq U, Y \subseteq V$, we can denote

$$\underline{R}'_U(X)(y) = \sup\{\alpha | y \in \underline{R}_U^{1-\alpha}(X)\}, \quad \overline{R}'_U(X)(y) = \sup\{\alpha | y \in \overline{R}_U^\alpha(X)\},$$
$$\underline{R}'_V(Y)(x) = \sup\{\alpha | x \in \underline{R}_V^{1-\alpha}(Y)\}, \quad \overline{R}'_V(Y)(x) = \sup\{\alpha | x \in \overline{R}_V^\alpha(Y)\},$$
$$\underline{R}''_U(X)(y) = \sup\{\alpha | y \in \underline{R}_U^{(1-\alpha)_+}(X)\}, \quad \overline{R}''_U(X)(y) = \sup\{\alpha | y \in \overline{R}_U^{\alpha_+}(X)\},$$
$$\underline{R}''_V(Y)(x) = \sup\{\alpha | x \in \underline{R}_V^{(1-\alpha)_+}(Y)\}, \quad \overline{R}''_V(Y)(x) = \sup\{\alpha | x \in \overline{R}_V^{\alpha_+}(Y)\},$$

where $\underline{R}'_U(X), \overline{R}'_U(X)(y)$ are called the fuzzy strong lower and upper approximation of the set X on the universe U, $\underline{R}'_V(Y)$, $\overline{R}'_V(Y)$ are named the fuzzy strong lower and upper approximation of the set Y on the universe V, $\underline{R}''_U(X)$, $\overline{R}''_U(X)$ are called the fuzzy weak lower and upper approximation of the set X on the universe U, and $\underline{R}''_V(Y)$, $\overline{R}''_V(Y)$ are named the fuzzy weak lower and upper approximation of the set Y on the universe V.

Theorem 3.13 *Let (U, V, R) be a fuzzy approximation space, then the following properties hold.*

(1) $\overline{R}'_U(X) = \overline{R}_U(X), \quad \overline{R}'_V(Y) = \overline{R}_V(Y);$

(2) $\underline{R}''_U(X) = \underline{R}_U(X), \quad \underline{R}''_V(Y) = \underline{R}_V(Y);$

(3) $\overline{R}'_U(X) = \sim\underline{R}''_U(\sim X), \quad \overline{R}'_V(Y) = \sim\underline{R}''_V(\sim Y).$

Proof According to the similarity of the above properties, we only need to prove the first part of each property.

(1) For any $y \in V$, denote
$$\beta_1 = \overline{R}_U(X)(y) = \max_{x \in X} R(x,y),$$
$$\beta_2 = \overline{R}'_U(X)(y) = \sup\{\alpha | R_p^\alpha(y) \cap X \neq \emptyset\};$$
according to the definition of β_1, we can know that there exists $x \in X$ s.t. $R(x, y) = \beta_1$, that is to say $x \in R_p^\alpha(y)$. So $R_p^\alpha(y) \cap X \neq \emptyset$, thus $\beta_2 \geq \beta_1$.
Suppose $\beta_2 > \beta_1$, we can see that there exists β_0 satisfying $\beta_2 > \beta_0 > \beta_1$, so we can obtain that there exists $x \in X$, s.t. $R(x,y) \geq \beta_0$ from the definition of β_2. Therefore, $\max_{x \in X} R(x,y) \geq \beta_0$, which isn't consistent with $\beta_0 > \beta_1$. Hence, $\beta_2 = \beta_1$.

(2) It is similar to prove that $\underline{R}''_U(X) = \underline{R}_U(X)$.

(3) By the items (1) and (2) we can have
$$\overline{R}'_U(X) = \overline{R}_U(X) = \sim\underline{R}_U(\sim X) = \sim\underline{R}''_U(\sim X).$$

Definition 3.9 Let R is a binary fuzzy relation on $U \times V$,

(1) R is a fuzzy serial relation, if for any $x \in U$ there exists $y \in V$ such that $R(x, y) = 1$,

(2) R is a fuzzy reverse serial relation, if for any $y \in V$ there exists $x \in U$ such that $R(x,y) = 1$.

Theorem 3.14 *Let (U, V, R) be a fuzzy approximation space, $\forall X \subseteq U, Y \subseteq V$, if R is a fuzzy serial relation, the $\overline{R}_U(X)$ is a normal fuzzy set, if R is a fuzzy reverse serial relation, the $\overline{R}_V(Y)$ is a normal fuzzy set.*

Proof Since R is a fuzzy serial relation, then for any $x \in U$ there exists $y \in V$, such that $R(x,y) = 1$. Thus, $\overline{R}_U(X)(y) = \max_{x \in X} R(x,y) = 1$.

Since R is a fuzzy reverse serial relation, then for any $y \in V$ there exists $x \in U$ such that $R(x,y) = 1$. Thus, $\overline{R}_V(Y)(x) = \max_{y \in Y} R(x,y) = 1$.

Theorem 3.15 *Let (U, V, R) be a fuzzy approximation space, for any $X \subseteq U, Y \subseteq V$, if R is a fuzzy reverse serial relation, then*

$\underline{R}_U(X) \subseteq \overline{R}_U(X)$;

if R is a fuzzy serial relation, then

$\underline{R}_V(Y) \subseteq \overline{R}_V(Y)$.

Proof Since R is a fuzzy reverse serial relation, then for any $y \in V$, there exists $x \in U$ such that $R(x,y) = 1$. Thus, for any $y \in V$ there exists $x \in U$ such that $\max_{x \in X} R(x,y) + \max_{x \notin X} R(x,y) \geq 1$, That is to say

$$\begin{aligned}\overline{R}_U(X)(y) - \underline{R}_U(X)(y) &= \max_{x \in X} R(x,y) - \min_{x \notin X}(1 - R(x,y)) \\ &= \max_{x \in X} R(x,y) + \max_{x \notin X} R(x,y) - 1 \\ &\geq 0.\end{aligned}$$

Thus, $\forall y \in V$, $\overline{R}_U(X)(y) \geq \underline{R}_U(X)(y)$, i.e., $\underline{R}_U(X) \subseteq \overline{R}_U(X)$.

If R is a fuzzy serial relation, then, $\forall\, x \in U$ there exists $y \in V$ such that $R(x,y) = 1$. Thus $\forall\, x \in U$ there exists $y \in V$ such that

$$\max_{y \in Y} R(x,y) + \max_{y \notin Y} R(x,y) \geq 1.$$

That is to say

$$\begin{aligned}\overline{R}_V(Y)(x) - \underline{R}_V(Y)(x) &= \max_{y \in Y} R(x,y) - \min_{y \notin Y}(1 - R(x,y)) \\ &= \max_{y \in Y} R(x,y) + \max_{y \notin Y} R(x,y) - 1 \\ &\geq 0.\end{aligned}$$

Thus, $\forall x \in U$, $\overline{R}_V(Y)(x) \geq \underline{R}_V(Y)(x)$, i.e., $\underline{R}_V(Y) \subseteq \overline{R}_V(Y)$.

3.3 Rough fuzzy set in a fuzzy approximation space

Definition 3.10 Let (U, V, R) is a fuzzy approximation space, i.e, R is a fuzzy relation on $U \times V$, for any $A \in F(U)$ $B \in F(V)$, the upper and lower approximations of A and B about (U, V, R), denoted by $\underline{R}_U A$, $\overline{R}_U A$, $\underline{R}_V B$ and $\overline{R}_V B$ are fuzzy sets and are, respectively, defined as follows:

$$\begin{aligned}&\underline{R}_U(A)(y) = \wedge\{A(x) \vee (1 - R(x,y)) | x \in U\} \quad y \in V,\\ &\overline{R}_U(A)(y) = \vee\{A(x) \wedge R(x,y) | x \in U\} \quad y \in V;\\ &\underline{R}_V(B)(x) = \wedge\{B(y) \vee (1 - R(x,y)) | y \in V\} \quad x \in U,\\ &\overline{R}_V(B)(x) = \vee\{B(y) \wedge R(x,y) | y \in V\} \quad x \in U.\end{aligned}$$

In the following, we employ an example to illustrate the above concepts.

Example 3.4 (Continued from Example 3.2) In Example 3.2, we have solved some questions, but we also can raise the other questions such as:

If the comprehensive scholarship of a student is $A = 0.5$, 0.3, 0.2, how about the evaluation of the student? And if we have got the evaluation of a student is $B =$ 0.4, 0.2, 0.2, 0.05, what is the comprehensive scholarship according to the student comprehensive evaluation system?

Now, we can solve the above questions according to Definition 3.10, we can have

$$\begin{aligned}&\underline{R}_U(A) = 0.3, 0.3, 0.3, 0.3;\\ &\overline{R}_U(A) = 0.5, 0.5, 0.5, 0.5;\\ &\underline{R}_V(B) = 0.2, 0.2, 0.3;\\ &\overline{R}_V(B) = 0.4, 0.4, 0.4.\end{aligned}$$

Remark 3.4 In a fuzzy approximation space, we can find out that the lower and upper approximations of fuzzy set $A \in F(U)$ belong to $F(V)$, and the lower and upper approximations of fuzzy set $B \in F(V)$ belong to $F(U)$ according the results of Example 3.4. This property is different from the lower and upper approximations over a universe.

Theorem 3.16 *Let* (U, V, R) *is a fuzzy approximation space, i.e, R is a fuzzy relation on* $U \times V$, *for any* $A, A' \in F(U)$, B, $B' \in F(V)$, *we have the following properties.*

(1) $\underline{R}_U(A) = \sim\overline{R}_U(\sim A), \underline{R}_V(B) = \sim\overline{R}_V(\sim B)$;

(2) $\underline{R}_U(A \cup \hat{\alpha}) = \underline{R}_U(A) \cup \hat{\alpha}$, $\overline{R}_U(A \cap \hat{\alpha}) = \overline{R}_U(A) \cap \hat{\alpha}$;
$\underline{R}_V(B \cup \hat{\alpha}) = \underline{R}_V(B) \cup \hat{\alpha}$, $\overline{R}_V(B \cap \hat{\alpha}) = \overline{R}_V(B) \cap \hat{\alpha}$;

(3) $\underline{R}_U(A \cap A') = \underline{R}_U(A) \cap \underline{R}_U(A')$,
$\overline{R}_U(A \cup A') = \overline{R}_U(A) \cup \overline{R}_U(A')$;
$\underline{R}_V(B \cap B') = \underline{R}_V(B) \cap \underline{R}_V(B')$,
$\overline{R}_V(B \cup B') = \overline{R}_V(B) \cup \overline{R}_V(B')$;

(4) $\underline{R}_U(A \cup A') \supseteq \underline{R}_U(A) \cup \underline{R}_U(A')$,
$\overline{R}_U(A \cap A') \subseteq \overline{R}_U(A) \cap \overline{R}_U(A')$;
$\underline{R}_V(B \cup B') \supseteq \underline{R}_V(B) \cup \underline{R}_V(B')$,
$\overline{R}_V(B \cap B') \subseteq \overline{R}_V(B) \cap \overline{R}_V(B')$.

Proof All terms can be proved by Definition 3.10.

Theorem 3.17 *Let* (U, V, R) *is fuzzy approximation space, i.e, R is a fuzzy relation on* $U \times V$, *for any* $A \in F(U)$, $B \in F(V)$, *we have*

$$\begin{aligned}\overline{R}_U(A) &= \bigvee_{\alpha \in [0,1]} [\alpha \wedge \overline{R^{\alpha}_U}(A_{\alpha})] = \bigvee_{\alpha \in [0,1]} [\alpha \wedge \overline{R^{\alpha}_U}(A_{\alpha_+})] \\ &= \bigvee_{\alpha \in [0,1]} [\alpha \wedge \overline{R^{\alpha_+}_U}(A_{\alpha})] = \bigvee_{\alpha \in [0,1]} [\alpha \wedge \overline{R^{\alpha_+}_U}(A_{\alpha_+})];\\ \overline{R}_V(B) &= \bigvee_{\alpha \in [0,1]} [\alpha \wedge \overline{R^{\alpha}_V}(B_{\alpha})] = \bigvee_{\alpha \in [0,1]} [\alpha \wedge \overline{R^{\alpha}_V}(B_{\alpha_+})] \\ &= \bigvee_{\alpha \in [0,1]} [\alpha \wedge \overline{R^{\alpha_+}_V}(B_{\alpha})] = \bigvee_{\alpha \in [0,1]} [\alpha \wedge \overline{R^{\alpha_+}_V}(B_{\alpha_+})].\end{aligned}$$

Proof For any $y \in V$,

$$\begin{aligned}\bigvee_{\alpha \in [0,1]} [\alpha \wedge \overline{R^{\alpha}_U}(A_{\alpha})] &= \sup\{\alpha \in [0,1] | x \in \overline{R^{\alpha}_U}(A_{\alpha})\} \\ &= \sup\{\alpha \in [0,1] | x \in R^{\alpha}_p(y) \cap (A_{\alpha}) \neq \emptyset\} \\ &= \sup\{\alpha \in [0,1] | \exists x \in U, x \in R^{\alpha}_p(y), x \in A_{\alpha}\} \\ &= \sup\{\alpha \in [0,1] | \exists x \in U, R(x,y) \geq \alpha, A(x) \geq \alpha\} \\ &= \vee A(x) \wedge R(x,y) | x \in U.\end{aligned}$$

So $\overline{R}_U(A) = \bigvee_{\alpha \in [0,1]} [\alpha \wedge \overline{R^{\alpha}_U}(A_{\alpha})]$.

The others of the equivalents can be proved similarly.

Theorem 3.18 *Let (U, V, R) is fuzzy approximation space, i.e, R is a fuzzy relation on U × V, for any* $A \in F(U)$, $B \in F(V)$, *we have*

$$\underline{R}_U(A) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \underline{R_U^{1-\alpha}}(A_\alpha)] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \underline{R_U^{1-\alpha}}(A_{\alpha_+})]$$
$$= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \underline{R_U^{(1-\alpha)_+}}(A_\alpha)] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \underline{R_U^{(1-\alpha)_+}}(A_{\alpha_+})];$$
$$\underline{R}_V(B) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \underline{R_V^{1-\alpha}}(B_\alpha)] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \underline{R_V^{1-\alpha}}(B_{\alpha_+})]$$
$$= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \underline{R_V^{(1-\alpha)_+}}(B_\alpha)] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \underline{R_V^{(1-\alpha)_+}}(B_{\alpha_+})].$$

Proof The proof is similar to Theorem 3.17.

Theorem 3.19 *Let (U, V, R) is fuzzy approximation space, i.e, R is a fuzzy relation on U × V, for any* $x \in U$, $y \in V$, $A \in F(U)$, $B \in F(V)$, *we have*

$$\overline{R}_U(A)(y) = \sup\{\alpha \in [0,1] | y \in \overline{R_U^{\alpha}}(A_\alpha)\}$$
$$= \sup\{\alpha \in [0,1] | y \in \overline{R_U^{\alpha_+}}(A_\alpha)\}$$
$$= \sup\{\alpha \in [0,1] | y \in \overline{R_U^{\alpha}}(A_{\alpha_+})\}$$
$$= \sup\{\alpha \in [0,1] | y \in \overline{R_U^{\alpha_+}}(A_{\alpha_+})\};$$
$$\overline{R}_V(B)(x) = \sup\{\alpha \in [0,1] | x \in \overline{R_V^{\alpha}}(B_\alpha)\}$$
$$= \sup\{\alpha \in [0,1] | x \in \overline{R_V^{\alpha_+}}(B_\alpha)\}$$
$$= \sup\{\alpha \in [0,1] | x \in \overline{R_V^{\alpha}}(B_{\alpha_+})\}$$
$$= \sup\{\alpha \in [0,1] | x \in \overline{R_V^{\alpha_+}}(B_{\alpha_+})\};$$
$$\underline{R}_U(A)(y) = \sup\{\alpha \in [0,1] | y \in \underline{R_U^{1-\alpha}}(A_\alpha)\}$$
$$= \sup\{\alpha \in [0,1] | y \in \underline{R_U^{(1-\alpha)_+}}(A_\alpha)\}$$
$$= \sup\{\alpha \in [0,1] | y \in \underline{R_U^{1-\alpha}}(A_{\alpha_+})\}$$
$$= \sup\{\alpha \in [0,1] | y \in \underline{R_U^{(1-\alpha)_+}}(A_{\alpha_+})\};$$
$$\underline{R}_V(B)(x) = \sup\{\alpha \in [0,1] | x \in \underline{R_V^{1-\alpha}}(B_\alpha)\}$$
$$= \sup\{\alpha \in [0,1] | x \in \underline{R_V^{(1-\alpha)_+}}(B_\alpha)\}$$
$$= \sup\{\alpha \in [0,1] | x \in \underline{R_V^{1-\alpha}}(B_{\alpha_+})\}$$
$$= \sup\{\alpha \in [0,1] | x \in \underline{R_V^{(1-\alpha)_+}}(B_{\alpha_+})\}.$$

Proof We can prove it by Theorems 3.17, 3.18.

4 The measures of fuzzy rough set models over two universes

In this section, we will research some measures of the fuzzy rough set over different universes.

Definition 4.1 Let (U, V, R) be a generalized approximation space, $A \in F(U)$, for any $0 \le \beta \le \alpha \le 1$, the approximate precision $\alpha_U(A)(\alpha, \beta)$ of A about R_U can be defined as following:

$$\alpha_U(A)(\alpha,\beta) = \frac{|(\underline{R}_U A)_\alpha|}{|(\overline{R}_U A)_\beta|},$$

where $A \neq \emptyset$, $|\cdot|$ denotes the cardinality of set.

Let $\rho_U(A)(\alpha,\beta) = 1 - \alpha_{R_U}(A)(\alpha,\beta)$, and $\rho_U(A)(\alpha,\beta)$ is called the rough degree of A about the universe U.

Theorem 4.1 *Let (U, V, R) be a generalized approximation space,* $A \in F(U)$, *for any* $0 \le \beta \le \alpha \le 1$, *the approximate precision* $\alpha_U(A)(\alpha, \beta)$ *and the rough degree* $\rho_U(A)(\alpha, \beta)$ *satisfy the properties as following:*

$$0 \le \alpha_U(A)(\alpha,\beta) \le 1, \quad 0 \le \rho_U(A)(\alpha,\beta) \le 1.$$

Proof According to Definition 4.1, this theorem can be proved easily.

Theorem 4.2 *Let (U, V, R) be a generalized approximation space,* $A, A_1 \in F(U)$, $A \subseteq A_1$, *and* $(\overline{R}_U A)_\beta = (\overline{R}_U A_1)_\beta$, *for any* $0 \le \beta \le \alpha \le 1$,

$$\alpha_U(A)(\alpha,\beta) \le \alpha_U(A_1)(\alpha,\beta), \quad \rho_U(A_1)(\alpha,\beta) \le \rho_U(A)(\alpha,\beta).$$

Proof Since $A \subseteq A_1$, we can have $(\underline{R}_U A)_\alpha \subseteq (\underline{R}_U A_1)_\alpha$. On the other hand, $(\overline{R}_U A)_\beta = (\overline{R}_U A_1)_\beta$. Therefore, the theorem can be proved by Definition 4.1.

Theorem 4.3 *Let (U, V, R) be a generalized approximation space,* $A \in F(U)$, $A_1 \in F(U)$, *and* $A \subseteq A_1$, $(\underline{R}_U A)_\alpha = (\underline{R}_U A_1)_\alpha$, *for any* $0 \le \beta \le \alpha \le 1$,

$$\alpha_U(A_1)(\alpha,\beta) \le \alpha_U(A)(\alpha,\beta), \quad \rho_U(A)(\alpha,\beta) \le \rho_U(A_1)(\alpha,\beta).$$

Proof The proof is similar to Theorem 4.2.

Theorem 4.4 *Let* $A, A_1 \in F(U)$, *if* $A_1 \approx_U A$, *for any* $0 \le \beta \le \alpha \le 1$, *we can have*

$$\alpha_U(A)(\alpha,\beta) = \alpha_U(A_1)(\alpha,\beta),$$
$$\rho_U(A_1)(\alpha,\beta) = \rho_U(A)(\alpha,\beta).$$

Proof It can be proved by Definition 3.4 and Definition 4.1.

Theorem 4.5 *Let U, V be two non-empty finite universes, R be the relation of U × V. For any* $A, A_1 \in F(U)$. *The rough degree and precision of the* A, A_1, $A \cup A_1$ *and* $A \cap A_1$ *satisfying the following relations.*

$$\rho_U(A\cup A_1)(\alpha,\beta)|(\overline{R}_U(A))_\beta \cup (\overline{R}_U(A_1))_\beta|$$
$$\le \rho_U(A)(\alpha,\beta)|(\overline{R}_U(A))_\beta| + \rho_U(A_1)(\alpha,\beta)|(\overline{R}_U(A_1))_\beta|$$
$$- \rho_U(A\cap A_1)(\alpha,\beta)|(\overline{R}_U(A))_\beta \cap (\overline{R}_U(A_1))_\beta|$$
$$\alpha_U(A\cup A_1)(\alpha,\beta)|(\overline{R}_U(A))_\beta \cup (\overline{R}_U(A_1))_\beta|$$
$$\ge \alpha_U(A)(\alpha,\beta)|(\overline{R}_U(A))_\beta| + \alpha_U(A_1)(\alpha,\beta)|(\overline{R}_U(A_1))_\beta|$$
$$- \alpha_U(A\cap A_1)(\alpha,\beta)|(\overline{R}_U(A))_\beta \cap (\overline{R}_U(A_1))_\beta|.$$

Proof According to Theorem 3.1, we can obtain

$$\rho_U(A\cup A_1)(\alpha,\beta) = 1-\frac{|(\underline{R}_U(A\cup A_1))_\alpha|}{|(\overline{R}_U(A\cup A_1))_\beta|}$$
$$= 1-\frac{|(\underline{R}_U(A\cup A_1))_\alpha|}{|(\overline{R}_U(A))_\beta\cup(\overline{R}_U(A_1))_\beta|}$$
$$\le 1-\frac{|(\underline{R}_U(A))_\alpha\cup(\underline{R}_U(A_1))_\alpha|}{|(\overline{R}_U(A))_\beta\cup(\overline{R}_U(A_1))_\beta|};$$

and

$$\rho_U(A\cap A_1)(\alpha,\beta) = 1-\frac{|(\underline{R}_U(A\cap A_1))_\alpha|}{|(\overline{R}_U(A\cap A_1))_\beta|}$$
$$= 1-\frac{|(\underline{R}_U(A))_\alpha\cap(\underline{R}_U(A_1))_\alpha|}{|(\overline{R}_U(A\cup A_1))_\beta|}$$
$$\le 1-\frac{|(\underline{R}(_U A))_\alpha\cap(\underline{R}_U(A_1))_\alpha|}{|(\overline{R}_U(A))_\beta\cap(\overline{R}_U(A_1))_\beta|}.$$

Hence,

$$\rho_U(A\cup A_1)(\alpha,\beta)|(\overline{R}_U(A))_\beta\cup(\overline{R}_U(A_1))_\beta|$$
$$\le |(\overline{R}_U(A))_\beta\cup(\overline{R}_U(A_1))_\beta| - |(\underline{R}_U(A))_\alpha\cup(\underline{R}_U(A_1))_\alpha|$$
$$= |(\overline{R}_U(A))_\beta| + |(\overline{R}_U(A_1))_\beta - |(\overline{R}_U(A))_\beta\cap(\overline{R}_U(A_1))_\beta|$$
$$- |(\underline{R}_U(A))_\alpha| - |(\underline{R}_U(A_1))_\alpha + |(\underline{R}_U(A))_\alpha\cap(\underline{R}_U(A_1))_\alpha|$$
$$\le |(\overline{R}_U(A))_\beta| + |(\overline{R}_U(A_1))_\beta| - |(\underline{R}_U(A))_\alpha| - |(\underline{R}_U(A_1))_\alpha|$$
$$\rho_U(A\cap A_1)(\alpha,\beta)|(\overline{R}_U(A))_\beta\cap(\overline{R}_U(A_1))_\beta|$$
$$= \rho_U(A)(\alpha,\beta)|(\overline{R}_U(A))_\beta| + \rho_U(A_1)(\alpha,\beta)|(\overline{R}_U(A_1))_\beta|$$
$$- \rho_U(A\cap A_1)(\alpha,\beta)|(\overline{R}_U(A))_\beta\cap(\overline{R}_U(A_1))_\beta|.$$

The other inequality can be proved similarly.

In the following, we will give some results about the measures of fuzzy approximation space.

Let (U, V, R) be a fuzzy approximation space, $g: U\mapsto V_d$, V_d is a nonempty finite integer set, d is the decision set, we call the (U, V, R, g, d) is a fuzzy decision approximation space. denote

$$D_k = x\in U|g(x) = k, \quad k\in V_d.$$

For any $\alpha\in[0,1]$, denote

$$R_a = \{(x,y)|R(x,y)\ge\alpha\},$$

then (U, V, R_α) is a generally approximation space, so we can have

$$U/R_\alpha = \{[x]_\alpha|x\in U\},$$

where

$$[x]_\alpha = \{x_i|R_s^\alpha(x_i) = R_s^\alpha(x)\},$$

the lower approximation of D_k in generally approximation space (U, V, R_α) can be defined as

$$\underline{R}_\alpha(D_k) = \{x\in U|[x]_\alpha\subseteq D_k\}, k\le r = |\ V_d\ |\ .$$

Definition 4.2 Let (U, V, R, g, d) be a fuzzy decision approximation space, $g: U\mapsto V_d$, denote

$$D(d/\alpha) = \frac{1}{|U|}\sum_{k=1}^{r}|\underline{R}_\alpha(D_k)|,$$

as the decision precision of α level. Denote

$$D'(d/\alpha) = \min\left\{\frac{|\underline{R}_\alpha(D_k)|}{|D_k|}: |k\le r\right\},$$

as the rule precision of α level.

Obviously, $D(d/\alpha)\ge D'(d/\alpha)$.

Theorem 4.6 *Let (U, V, R, g, d) be a fuzzy decision approximation space, if $\alpha<\beta$, then*

$$D(d/\alpha)\ge D(d/\beta), D'(d/\alpha)\ge D'(d/\beta).$$

Proof Since $\alpha<\beta$, we can have $R_\beta\subseteq R_\alpha$ and $[x]_\beta\subseteq[x]_\alpha$, so we can have

$$\underline{R}_\alpha(D_k)\subseteq\underline{R}_\beta(D_k)(\forall k\le l).$$

According to Definition 4.1 we can obtain

$$D(d/\alpha)\ge D(d/\beta), D'(d/\alpha)\ge D'(d/\beta).$$

Definition 4.3 Let (U, V, R) and (U, V, G) be two fuzzy approximation spaces, then (U, V, R, G) is called fuzzy objective approximation space.

Let (U, V, R, G) be a fuzzy objective approximation space, for any $0\le\alpha,\beta\le 1$ denote

$$R_\alpha = \{(x,y)\in U\times V|R(x,y)\ge\alpha\},$$
$$G_\beta = \{(x,y)\in U\times V|G(x,y)\ge\beta\}.$$

So we have

$$U/R_\alpha = \{[x]_\alpha|x\in U\},\quad [x]_\alpha = \{x_i|R_s^\alpha(x_i) = R_s^\alpha(x)\},$$
$$U/G_\beta = \{[x]_\beta|x\in U\},\quad [x]_\beta = \{x_i|G_s^\beta(x_i) = G_s^\beta(x)\},$$

where

$$R_s^\alpha(x) = \{y\in V|(x,y)\in R_\alpha\},$$
$$G_s^\beta(x) = \{y\in V|(x,y)\in G_\beta\}.$$

Definition 4.4 Let (U, V, R, G) be a fuzzy objective approximation space, for any $0\le\alpha,\beta\le 1$, denote

$$D_U^\beta(d/\alpha) = \frac{1}{|U|}\sum_{D\in U/G_\beta}|\underline{R}_\alpha(D)|,$$

as the decision precision of (α, β) level, denote

$$D_U'^\beta(d/\alpha) = \min\frac{|\underline{R}_\alpha(D)}{|D|}||D\in U/G_\beta,$$

as the rule precision of (α, β) level, where

$\underline{R}_U^{\alpha}(D) = \{x \in U | [x]_{\alpha} \subseteq D\}$.

Theorem 4.7 *Let* (U, V, R, G) *be a fuzzy objective approximation space, for any* $0 \leq \alpha, \beta \leq 1$, *we have*

(1) $D_U^{\beta}(d/\alpha) \geq D_U'^{\beta}(d/\alpha)$,

(2) $\alpha_1 < \alpha_2 \Rightarrow D_U^{\beta}(d/\alpha_1) \leq D_U^{\beta}(d/\alpha_2)$, $D_U'^{\beta}(d/\alpha_1) \leq D_U'^{\beta}(d/\alpha_2)$,

(3) $\beta_1 < \beta_2 \Rightarrow D_U^{\beta_2}(d/\alpha) \leq D_U^{\beta_1}(d/\alpha)$, $D_U'^{\beta_2}(d/\alpha) \leq D_U'^{\beta_1}(d/\alpha)$.

Proof According to Definition 4.3 and 4.4, the properties can be proved easily.

Remark 4.1 The results about the universe U also hold for the universe V.

5 The relationships among fuzzy rough set models and rough set model over two universes

Theorem 5.1 *If* $A \subseteq U$, $B \subseteq V$ *and* $R \subseteq U \times V$, *then the rough fuzzy set in fuzzy approximation space can be degenerated into rough set model over two universes.*

Proof According to the definitions of rough fuzzy set in fuzzy approximation space, we can have

$$\begin{aligned}
\underline{R}_U(A)(y) = 1 &\Leftrightarrow \forall x \in U, A(x) \vee (1 - R(x,y)) = 1 \\
&\Leftrightarrow \forall x \in U, \text{ if } x \notin A \Rightarrow (x,y) \notin R \\
&\Leftrightarrow \forall x \notin A \Rightarrow x \notin R_p(y) \\
&\Leftrightarrow R_p(y) \subseteq A, \\
\overline{R}_U(A)(y) = 1 &\Leftrightarrow \exists x \in U, s.t., A(x) = 1 \quad and \quad R(x,y) \\
&= 1 \Leftrightarrow A \cap R_p(y) \neq \emptyset, \\
\underline{R}_V(B)(x) = 1 &\Leftrightarrow \forall y \in V, B(y) \vee (1 - R(x,y)) = 1 \\
&\Leftrightarrow \forall y \in V, \text{ if } y \notin B \Rightarrow (x,y) \notin R \\
&\Leftrightarrow \forall x \notin B \Rightarrow y \notin R_s(x) \\
&\Leftrightarrow R_s(x) \subseteq B, \\
\overline{R}_V(B)(x) = 1 &\Leftrightarrow \exists y \in V, s.t., B(y) = 1 \quad and \\
&\quad R(x,y) = 1 \Leftrightarrow B \cap R_s(x) \neq \emptyset.
\end{aligned}$$

Theorem 5.2 *If* $A \in F(U)$, $B \in F(V)$ *and* $R \subseteq U \times V$, *then rough fuzzy set in a fuzzy approximation space can be degenerated into rough fuzzy set in a generalized approximation space.*

Proof According to the definitions of rough fuzzy set in fuzzy approximation space, we can have

$$\begin{aligned}
\underline{R}_U(A)(y) &= \min\{A(x) \vee (1 - R(x,y)) | x \in U\} \\
&= \min\{A(x) | (x,y) \in R\} \\
&= \min\{A(x) | x \in R_p(y)\}, \\
\overline{R}_U(A)(y) &= \max\{A(x) \wedge R(x,y) | x \in U\} \\
&= \max\{A(x) | (x,y) \in R\} \\
&= \max\{A(x) | x \in R_p(y)\}, \\
\underline{R}_V(B)(x) &= \min\{B(y) \vee (1 - R(x,y)) | y \in V\} \\
&= \min\{B(y) | (x,y) \in R\} = \min\{B(y) | y \in R_s(x)\}, \\
\overline{R}_V(B)(x) &= \max\{B(y) \wedge R(x,y) | y \in V\} \\
&= \max\{B(y) | (x,y) \in R\} \\
&= \max\{B(y) | y \in R_s(x)\}.
\end{aligned}$$

Theorem 5.3 *If* $A \subseteq U$, $B \subseteq V$ *and* $R : U \times V \mapsto [0,1]$, *then rough fuzzy set in a fuzzy approximation space can be degenerated rough set in a fuzzy approximation space.*

Proof According to the definitions of rough fuzzy set in fuzzy approximation space, we can have

$$\begin{aligned}
\underline{R}_U(A)(y) &= \min\{A(x) \vee (1 - R(x,y)) | x \in U\} \\
&= \min_{x \notin A}(1 - R(x,y)), \\
\overline{R}_U(A)(y) &= \max\{A(x) \wedge R(x,y) | x \in U\} = \max_{x \in A} R(x,y), \\
\underline{R}_V(B)(x) &= \min\{B(y) \vee (1 - R(x,y)) | y \in V\} \\
&= \min_{y \notin B}(1 - R(x,y)), \\
\overline{R}_V(B)(x) &= \max\{B(y) \wedge R(x,y) | y \in V\} = \max_{y \in B} R(x,y).
\end{aligned}$$

Theorem 5.4 *If* $A \subseteq U$, $B \subseteq V$ *and* $R \subseteq U \times V$, *then the rough fuzzy set in a generalized approximation space can be degenerated into rough set model over two universes.*

Proof According to the definitions of rough fuzzy set in a generalized approximation space, we can have

$$\begin{aligned}
&\underline{R}_U(A)(y) = 1 \Leftrightarrow \forall x \in R_p(y) \Rightarrow A(x) = 1 \Leftrightarrow R_p(y) \subseteq A, \\
&\overline{R}_U(A)(y) = 1 \Leftrightarrow \exists x \in R_p(y) \Rightarrow A(x) = 1 \Leftrightarrow R_p(y) \cap A \neq \emptyset, \\
&\underline{R}_V(B)(x) = 1 \Leftrightarrow \forall y \in R_s(x) \Rightarrow B(y) = 1 \Leftrightarrow R_s(x) \subseteq B, \\
&\overline{R}_V(B)(x) = 1 \Leftrightarrow \exists y \in R_s(x) \Rightarrow B(y) = 1 \Leftrightarrow R_s(x) \cap B \neq \emptyset.
\end{aligned}$$

Theorem 5.5 *If* $A \subseteq U$, $B \subseteq V$ *and* $R \subseteq U \times V$, *then the rough set in fuzzy approximation space can be degenerated into rough set model over two universes.*

Proof According to the definitions of rough set in fuzzy approximation space, we can have

$$\begin{aligned}
\underline{R}_U(A)(y) = 1 &\Leftrightarrow \forall x \notin A \Rightarrow R(x,y) = 0 \Leftrightarrow \forall x \notin A \Rightarrow x \notin R_p(y) \\
&\Leftrightarrow \forall x \in R_p(y) \Rightarrow x \in A \Leftrightarrow R_p(y) \subseteq A, \\
\overline{R}_U(A)(y) = 1 &\Leftrightarrow \exists x \in A \Rightarrow R(x,y) = 1 \Leftrightarrow \exists x \in A \Rightarrow x \in R_p(y) \\
&\Leftrightarrow R_p(y) \cap A \neq \emptyset, \\
\underline{R}_V(B)(x) = 1 &\Leftrightarrow \forall y \notin B \Rightarrow R(x,y) = 0 \Leftrightarrow \forall y \notin B \Rightarrow y \notin R_s(x) \\
&\Leftrightarrow \forall y \in R_s(x) \Rightarrow y \in B \Leftrightarrow R_s(x) \subseteq B, \\
\overline{R}_V(B)(x) = 1 &\Leftrightarrow \exists y \in B \Rightarrow R(x,y) = 1 \Leftrightarrow \exists y \in B \Rightarrow y \in R_s(x) \\
&\Leftrightarrow R_s(x) \cap B \neq \emptyset.
\end{aligned}$$

According to the Theorem 5.1, 5.2, 5.3,5.4,5.5 we can know that the rough set models over two universes are the special cases of the fuzzy rough set models over two universes. We also can easily find out that the fuzzy rough set models over two universes can be degenerated into Pawlak rough set model over a universe according to Remark 2.1.

6 Conclusion

From a generalized approximation space (U, V, R) to a fuzzy approximation space (U, V, R), we have defined rough fuzzy set in generalize approximation, rough set in fuzzy approximation space and rough fuzzy set in fuzzy approximation space over different universes which respectively resulted from the approximation of fuzzy information system on a universe, and mainly discussed the properties of the approximation operators and α level lower and upper approximation operators. In addition, the inter-relationships of the lower and upper approximation on α level of the fuzzy relation R and the α level of the lower and upper approximation of the fuzzy relation R are analyzed. What's more, the measures of the fuzzy rough set models are also studied. Finally, the relationships among fuzzy rough set models and rough set models over two universes are investigated.

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