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On Rough Membership Degrees in Ordered Information Systems

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Abstract: In this paper, we aim to consider roughness of a rough set in ordered information system, and the rough membership degree is proposed based on rough set theory. Moreover, some important properties of rough membership degree are discussed, and then an illustrative example is presented in ordered information system.

Keywords: Dominance relation; Ordered information system; Rough membership degree; Rough set

1. Introduction

The rough set theory, which is proposed by Pawlak in 1980s [6] for the study of intelligent systems, is characterized by uncertain or insufficient information. Nowadays, with the rapid development of the rough set, the rough set theory has been successfully applied in machine learning, pattern recognition, expert systems, data analysis, data mining and so on [7, 8, 12].

In Pawlak's original rough set theory, partition or equivalence relation is an important and primitive concept. And in the ordered information system, we mostly discuss the dominance relations, which induce generalized rough sets [1-4, 11, 13].

In this paper, we mainly study a measure of roughness in generalized rough sets of the ordered information system. This paper is organized as follows: In section 2, we recall some important concepts, such as Pawlak's rough sets and rough sets based on ordered information system. In section 3, we define the roughness of a rough set in the ordered information system, and investigate some properties of this measure. Lastly, we make brief conclusions in section 4.

2. Preliminaries

A Pawlak approximation space is an ordered pair $K=(U,R)$, where U is a non-empty and finite set of objects,

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called the universe and R is an equivalence relation on U.

Definition 2.1([6]) Let $K=(U,R)$ be a Pawlak approximation space, $[x]_R$ is the equivalence class of R, $X\subseteq U$, the sets

$$\underline{R}(X)=\{x\in U\mid [x]_R\subseteq X\}\quad \text{and}\quad \overline{R}(X)=\{x\in U\mid [x]_R\cap X\neq\phi\}$$

are, respectively, called the lower and upper approximation of X in (U,R).

If $\underline{R}(X)=\overline{R}(X)$, then X is a definable set, otherwise X is a rough set.

Rough set theory is a useful tool to express knowledge in information system. Here we recall the following definitions, which can be found in literatures [5, 9, 10].

Definition 2.2 The triple $I^{\geq}=(U,A,F)$ is called an ordered information system (OIS), where

(1) U is a non-empty finite set of objects, and $U=\{x_1,x_2,\cdots,x_n\}$;

(2) A is a non-empty and finite set of attributes, and $A=\{a_1,a_2,\cdots,a_m\}$;

(3) $F=\{f_l\mid U\to V_l, l\leq m\}$, $f_l(x)$ is the value of a_l on $x\in U$, V_l is the domain of a_l, $a_l\in A$;

(4) all attributes are criterions.

Definition 2.3 Let $I^{\geq}=(U,A,F)$ be an ordered information system, $R_A^{\geq}$ is a dominance relation in $I^{\geq}$, denote $R_A^{\geq}=\{(x,y)\in U\times U\mid f_l(x)\geq f_l(y), \forall a_l\in A\}$, and $U/R_A^{\geq}=\{[x]_{R_A^{\geq}}\mid x\in U\}$ is the set of dominance relation class about $R_A^{\geq}$, where $[x]_{R_A^{\geq}}$ is called the dominance class contains x, and $[x]_{R_A^{\geq}}=\{y\in U\mid (y,x)\in R_A^{\geq}\}$.

Definition 2.4 Let $I^{\geq}=(U,A,F)$ be an ordered information system, $X\subseteq U$, the lower and upper approximations of X in the ordered information system are defined in the following:

$$\underline{R}_A^{\geq}(X)=\{x\in U\mid [x]_{R_A^{\geq}}\subseteq X\}\quad \text{and}\quad \overline{R}_A^{\geq}(X)=\{x\in U\mid [x]_{R_A^{\geq}}\cap X\neq\phi\}.$$

If $\underline{R}_A^{\geq}(X)=\overline{R}_A^{\geq}(X)$, then X is a definable set, otherwise X is a rough set with respect to $I^{\geq}$.

Positive region, negative region and boundary region are defined as follows:

$$POS_{R_A^{\geq}}(X)=\underline{R}_A^{\geq}(X),\ NEG_{R_A^{\geq}}(X)=U-\overline{R}_A^{\geq}(X),\ BN_{R_A^{\geq}}(X)=\overline{R}_A^{\geq}(X)-\underline{R}_A^{\geq}(X).$$

The lower and upper approximations of X have the following properties**.**

(1) $\underline{R}_A^{\geq}(X)\subseteq X\subseteq\overline{R}_A^{\geq}(X)$;

(2) $\underline{R}_A^{\geq}(\sim X)=\sim\overline{R}_A^{\geq}(X),\ \overline{R}_A^{\geq}(\sim X)=\sim\underline{R}_A^{\geq}(X)$;

(3) $\underline{R}_A^{\geq}(\phi)=\phi,\ \overline{R}_A^{\geq}(\phi)=\phi,\ \underline{R}_A^{\geq}(U)=U,\ \overline{R}_A^{\geq}(U)=U$;

(4) $\underline{R}_A^{\geq}(X\cap Y)=\underline{R}_A^{\geq}(X)\cap\underline{R}_A^{\geq}(Y),\ \overline{R}_A^{\geq}(X\cup Y)=\overline{R}_A^{\geq}(X)\cup\overline{R}_A^{\geq}(Y)$;

(5) $\underline{R}_A^{\geq}(X\cup Y)\supseteq\underline{R}_A^{\geq}(X)\cup\underline{R}_A^{\geq}(Y),\ \overline{R}_A^{\geq}(X\cap Y)\subseteq\overline{R}_A^{\geq}(X)\cap\overline{R}_A^{\geq}(Y)$;

(6) $X\subseteq Y\Rightarrow\underline{R}_A^{\geq}(X)\subseteq\underline{R}_A^{\geq}(Y),\ X\subseteq Y\Rightarrow\overline{R}_A^{\geq}(X)\subseteq\overline{R}_A^{\geq}(Y)$;

(7) $\underline{R}_A^{\geq}(\underline{R}_A^{\geq}(X))=\underline{R}_A^{\geq}(X),\ \overline{R}_A^{\geq}(\overline{R}_A^{\geq}(X))=\overline{R}_A^{\geq}(X)$.

3. A roughness measure of rough sets in ordered information system

Let $I^{\geq}=(U,A,F)$ be an ordered information system, $R_A^{\geq}$ is a dominance relation in $I^{\geq}$, and $U/R_A^{\geq}=\{[x]_{R_A^{\geq}}\mid x\in U\}$ is the dominance relation class about $R_A^{\geq}$.

Definition 3.1 Let $I^{\geq}=(U,A,F)$ be an ordered information system, $X\subseteq U$, $R_A^{\geq}$ is a dominance relation in $I^{\geq}$, then the rough set is $(\underline{R}_A^{\geq}(X),\overline{R}_A^{\geq}(X))$, for any $x\in U$, rough membership degree of x in X with respect to $R_A^{\geq}$, denoted by $u_{R_A^{\geq}}^{X}(x)$, is defined by

$$u_{R_A^{\geq}}^{X}(x)=\frac{|[x]_{R_A^{\geq}}\cap X|}{|[x]_{R_A^{\geq}}|}.$$

Obviously, for any $x\in U$, $0\leq u_{R_A^{\geq}}^{X}(x)\leq 1$ holds.

In the next, we will discuss the properties of the rough membership of rough set in the ordered information system.

Proposition 3.1 Let $I^{\geq}=(U,A,F)$ be an ordered information system, and $U/R_A^{\geq}=\{[x]_{R_A^{\geq}}\mid x\in U\}$ is classification induced by a dominance relation $R_A^{\geq}$. The rough membership degree has the following properties:

(1) $u_{R_A^{\geq}}^{U}(x)=1$;

(2) $u_{R_A^{\geq}}^{\phi}(x)=0$;

(3) $u_{R_A^{\geq}}^{X}(x)=1\Leftrightarrow x\in POS_{R_A^{\geq}}(X)$;

(4) $u_{R_A^{\geq}}^{X}(x)=0\Leftrightarrow x\in NEG_{R_A^{\geq}}(X)$;

(5) $0 < u^X_{R^{\geq}_A}(x) < 1 \Leftrightarrow x \in BN_{R^{\geq}_A}(X)$;

(6) $u^{U-X}_{R^{\geq}_A}(x) = 1 - u^X_{R^{\geq}_A}(x)$.

Proof: (1) From the definition3.1, we can easily get

$$u^U_{R^{\geq}_A}(x) = \frac{|[x]_{R^{\geq}_A} \cap U|}{|[x]_{R^{\geq}_A}|} = \frac{|[x]_{R^{\geq}_A}|}{|[x]_{R^{\geq}_A}|} = 1.$$

(2) From the definition3.1, when $X = \phi$, we can obtain

$$u^{\phi}_{R^{\geq}_A}(x) = \frac{|[x]_{R^{\geq}_A} \cap \phi|}{|[x]_{R^{\geq}_A}|} = \frac{|\phi|}{|[x]_{R^{\geq}_A}|} = 0.$$

(3) From the above definitions, we can directly get

$$u^X_{R^{\geq}_A}(x) = 1 \Leftrightarrow [x]_{R^{\geq}_A} \subseteq X \Leftrightarrow x \in \underline{R}^{\geq}_A(X) \Leftrightarrow x \in POS_{R^{\geq}_A}(X).$$

(4) From the above definitions, we have

$$u^X_{R^{\geq}_A}(x) = 0 \Leftrightarrow [x]_{R^{\geq}_A} \cap X = \phi \Leftrightarrow x \notin \overline{R}^{\geq}_A(X) \Leftrightarrow x \in U - \overline{R}^{\geq}_A(X) \Leftrightarrow x \in NEG_{R^{\geq}_A}(X).$$

(5) It can be easily derived by (3) and (4).

(6) $u^{U-X}_{R^{\geq}_A}(x) = \dfrac{|[x]_{R^{\geq}_A} \cap (U-X)|}{|[x]_{R^{\geq}_A}|} = \dfrac{|[x]_{R^{\geq}_A} \cap U - [x]_{R^{\geq}_A} \cap X|}{|[x]_{R^{\geq}_A}|} = \dfrac{|[x]_{R^{\geq}_A}| - |[x]_{R^{\geq}_A} \cap X|}{|[x]_{R^{\geq}_A}|} = 1 - u^X_{R^{\geq}_A}(x).$

Then the proof is completed. □

From this proposition, we can find that $u^X_{R^{\geq}_A}(x)$ indicates degree of rough membership of x in X with respect to $R^{\geq}_A$.

Proposition 3.2 Let $I^{\geq} = (U, A, F)$ be an ordered information system, and $U / R^{\geq}_A = \{[x]_{R^{\geq}_A} \mid x \in U\}$ is classification induced by a dominance relation $R^{\geq}_A$. If X is a definable set of U, then

(1) $u^X_{R^{\geq}_A}(x) = 1$ if and only if $x \in X$;

(2) $u^X_{R^{\geq}_A}(x) = 0$ if and only if $x \notin X$.

Proof: (1) Since X is a definable set of U, we have $\underline{R}^{\geq}_A(X) = \overline{R}^{\geq}_A(X)$. From $\underline{R}^{\geq}_A(X) \subseteq X$, and $X \subseteq \overline{R}^{\geq}_A(X)$, we can get $\underline{R}^{\geq}_A(X) = X = \overline{R}^{\geq}_A(X)$. Accordingly, we can get the conclusions. If $x \in X$, then $u^X_{R^{\geq}_A}(x) = 1$. If $u^X_{R^{\geq}_A}(x) = 1$, then $x \in POS_{R^{\geq}_A}(X)$, hence $x \in X$.

(2) $u^X_{R^{\geq}_A}(x) = 0$, $[x]_{R^{\geq}_A} \cap X = \phi$, it equals to $x \notin \overline{R}^{\geq}_A(X)$. Since $X = \overline{R}^{\geq}_A(X)$, then $x \notin X$; if $x \notin X$, then $x \notin \overline{R}^{\geq}_A(X)$, and it equals to $[x]_{R^{\geq}_A} \cap X = \phi$, therefore $u^X_{R^{\geq}_A}(x) = 0$.

The proof is completed. □

Proposition 3.3 Let $I^{\geq}=(U,A,F)$ be an ordered information system, and $U/R_A^{\geq}=\{[x]_{R_A^{\geq}} \mid x\in U\}$ is classification induced by a dominance relation $R_A^{\geq}$. For any $X,Y\subseteq U$, the following conclusions hold.

(1) $X\subseteq Y \Rightarrow u_{R_A^{\geq}}^{X}(x)\leq u_{R_A^{\geq}}^{Y}(x)$;

(2) $u_{R_A^{\geq}}^{X\cup Y}(x)\geq MAX(u_{R_A^{\geq}}^{X}(x),u_{R_A^{\geq}}^{Y}(x))$;

(3) $u_{R_A^{\geq}}^{X\cap Y}(x)\leq MIN(u_{R_A^{\geq}}^{X}(x),u_{R_A^{\geq}}^{Y}(x))$;

(4) $u_{R_A^{\geq}}^{X\cup Y}(x)=u_{R_A^{\geq}}^{X}(x)+u_{R_A^{\geq}}^{Y}(x)-u_{R_A^{\geq}}^{X\cap Y}(x)$;

(5) $X\cap Y=\phi \Rightarrow u_{R_A^{\geq}}^{X\cup Y}(x)=u_{R_A^{\geq}}^{X}(x)+u_{R_A^{\geq}}^{Y}(x)$.

Proof: (1) As $X\subseteq Y$, we can get $[x]_{R_A^{\geq}}\cap X\subseteq [x]_{R_A^{\geq}}\cap Y$, so $u_{R_A^{\geq}}^{X}(x)\leq u_{R_A^{\geq}}^{Y}(x)$.

(2) Since $[x]_{R_A^{\geq}}\cap X\subseteq [x]_{R_A^{\geq}}\cap (X\cup Y)$ and $[x]_{R_A^{\geq}}\cap Y\subseteq [x]_{R_A^{\geq}}\cap (X\cup Y)$, we can obtain $u_{R_A^{\geq}}^{X\cup Y}(x)\geq$ $MAX(u_{R_A^{\geq}}^{X}(x),u_{R_A^{\geq}}^{Y}(x))$.

(3)According to $[x]_{R_A^{\geq}}\cap X\supseteq [x]_{R_A^{\geq}}\cap (X\cap Y)$ and $[x]_{R_A^{\geq}}\cap Y\supseteq [x]_{R_A^{\geq}}\cap (X\cap Y)$, we can get $u_{R_A^{\geq}}^{X\cap Y}(x)\leq$ $MIN(u_{R_A^{\geq}}^{X}(x),u_{R_A^{\geq}}^{Y}(x))$.

(4) From the definition, we have the following equations:

$$\begin{aligned} u_{R_A^{\geq}}^{X\cup Y}(x) &= \frac{|[x]_{R_A^{\geq}}\cap (X\cup Y)|}{|[x]_{R_A^{\geq}}|} \\ &= \frac{|([x]_{R_A^{\geq}}\cap X)|+|([x]_{R_A^{\geq}}\cap Y)|-|[x]_{R_A^{\geq}}\cap (X\cap Y)|}{|[x]_{R_A^{\geq}}|} \\ &= u_{R_A^{\geq}}^{X}(x)+u_{R_A^{\geq}}^{Y}(x)-u_{R_A^{\geq}}^{X\cap Y}(x). \end{aligned}$$

(5) Since $X\cap Y=\phi$, so this item can be proved according to (4). □

Proposition 3.4 Let $I^{\geq}=(U,A,F)$ be an ordered information system, and $U/R_A^{\geq}=\{[x]_{R_A^{\geq}} \mid x\in U\}$ is classification induced by a dominance relation $R_A^{\geq}$. If $X=\{X_1,X_2\cdots,X_N\}$ is a family of sets, which are disjoint to each other, then for any $x\in U$, $u_{R_A^{\geq}}^{\cup X_i}(x)=\sum_{X_i\in X} u_{R_A^{\geq}}^{X_i}(x)$ holds. Especially, if $X=\{X_1,X_2\cdots,X_N\}$ is a partition of U , then for any $x\in U$, $u_{R_A^{\geq}}^{\cup X_i}(x)=1$.

Proof: From the (4) and (5) in the proposition 3.3, we can get $u_{R_A^{\geq}}^{\cup X_i}(x)=\sum_{X_i\in X} u_{R_A^{\geq}}^{X_i}(x)$.

When $X=\{X_1,X_2\cdots,X_N\}$ is a partition of U , for any $x\in U$, we have

$$u_{R_A^{\geq}}^{X_1}(x)+u_{R_A^{\geq}}^{X_2}(x)+\cdots+u_{R_A^{\geq}}^{X_N}(x)$$

$$=\frac{|[x]_{R_A^{\geq}}\cap X_1|}{|[x]_{R_A^{\geq}}|}+\frac{|[x]_{R_A^{\geq}}\cap X_2|}{|[x]_{R_A^{\geq}}|}+\cdots+\frac{|[x]_{R_A^{\geq}}\cap X_N|}{|[x]_{R_A^{\geq}}|}$$

$$=\frac{|[x]_{R_A^{\geq}}\cap U|}{|[x]_{R_A^{\geq}}|}=1.$$

Then the proposition has been proved. □

Table 1 An ordered information system

U	a_1	a_2	a_3
x_1	1	1	2
x_2	1	2	1
x_3	1	2	3
x_4	2	1	3
x_5	3	1	2
x_6	3	2	2
x_7	2	2	1

Example 1 Let's consider the following ordered information system $I^{\geq}$ (Table 1), where the universe $U=\{x_1,x_2,\cdots,x_7\}$, and $A=\{a_1,a_2,a_3\}$. From Table 1, we can get:

$$[x_1]_{R_A^{\geq}}=\{x_1,x_3,x_4,x_5,x_6\},\ [x_2]_{R_A^{\geq}}=\{x_2,x_3,x_6,x_7\},\ [x_3]_{R_A^{\geq}}=\{x_3\},$$

$$[x_4]_{R_A^{\geq}}=\{x_4\},\ [x_5]_{R_A^{\geq}}=\{x_5,x_6\},\ [x_6]_{R_A^{\geq}}=\{x_6\},\ [x_7]_{R_A^{\geq}}=\{x_6,x_7\}.$$

If we take sets $X_1=\{x_1,x_3,x_4\}$ and $X_2=\{x_2,x_5,x_6,x_7\}$, then

$$\underline{R}_A^{\geq}(X_1)=\{x_3,x_4\},\ \overline{R}_A^{\geq}(X_1)=\{x_1,x_2,x_3,x_4\};$$

$$\underline{R}_A^{\geq}(X_2)=\{x_5,x_6,x_7\},\ \overline{R}_A^{\geq}(X_2)=\{x_1,x_2,x_5,x_6,x_7\}.$$

And the rough membership degrees of $x_i(x_i\in U)$ in both X_1 and X_2 with respect to $R_A^{\geq}$ are

$$u_{R_A^{\geq}}^{X_1}(x_1)=\frac{3}{5},\quad u_{R_A^{\geq}}^{X_2}(x_1)=\frac{2}{5};\ u_{R_A^{\geq}}^{X_1}(x_2)=\frac{1}{4},\quad u_{R_A^{\geq}}^{X_2}(x_2)=\frac{3}{4};$$

$$u_{R_A^{\geq}}^{X_1}(x_3)=1, \quad u_{R_A^{\geq}}^{X_2}(x_3)=0; \; u_{R_A^{\geq}}^{X_1}(x_4)=1, \quad u_{R_A^{\geq}}^{X_2}(x_4)=0;$$

$$u_{R_A^{\geq}}^{X_1}(x_5)=0, \quad u_{R_A^{\geq}}^{X_2}(x_5)=1; \; u_{R_A^{\geq}}^{X_1}(x_6)=0, \quad u_{R_A^{\geq}}^{X_2}(x_6)=1;$$

$$u_{R_A^{\geq}}^{X_1}(x_7)=0, \quad u_{R_A^{\geq}}^{X_2}(x_7)=1.$$

Here, as $\{X_1, X_2\}$ forms a partition of U , for each $x_i\,(x_i \in U)$, $u_{R_A^{\geq}}^{X_1}(x_i)+u_{R_A^{\geq}}^{X_2}(x_i)=1$ holds, then the above propositions can be tested.

4. Conclusions

The rough set has been a generalization of classical set theory, and it is an important method to deal with uncertainty. In this paper, we discussed the lower and upper approximations in the ordered information system and introduced a measure of roughness in the ordered information system. Accordingly, several main properties of this measure have been proposed.

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