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A novel rough set method based on adjustable-perspective dominance relations in intuitionistic fuzzy ordered decision tables

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ABSTRACT

With the increasing diversification and complexity of information and data, it is crucial to monitor and process data and information from multiple perspectives. There is a general consensus that dominance-based rough set approaches are the most effective methods for ordered information system research. Regardless, limitations and irrationality still exist in dominance relations because they cannot reflect the different emphases of data under different feature sets, nor can they meet the requirements for describing data information in the real world. To enable the dominance relation rough set model to be more effectively applied to practical problems in line with human cognition, our work focuses on developing adjustable-perspective dominance relations by fusing three different dominance relations in an intuitionistic fuzzy ordered decision table. On this basis, we construct a single-perspective rough set (SPRS) model to realize knowledge mining and rule extraction from various perspectives. Additionally, we present two types of different-perspective rough sets (DPRS) that reduce the restriction of single-perspective evaluation data in realistic problems and discuss rule extraction. Additionally, we compare SPRS and DPRS to other dominance-based rough set models from the perspectives of the ordinal classification, roughness, and dependence degree. Finally, we analyze eight UCI datasets and present a series of comparative experiments to demonstrate the effectiveness and rationality of the proposed model.

1. Introduction

As science and technology develop, the number and frequency of data are increasing at an unprecedented rate. It is worth noting that there are many uncertainty phenomena in massive data, which have significant value for the research of uncertainty theory. As a powerful tool for handling uncertainty, the rough set theory (RST) proposed by Pawlak [1] focuses on lower and upper approximations, and characterizes uncertain information based on known information. For handling different data types in information systems, scholars have proposed various extended models such as the neighborhood rough set [2], interval-valued rough set [3], fuzzy rough set [4,5], decision-theoretic rough set [6,7]. Considering the degree of information quantification and level of decision risk, Xu et al. applied three-way decision and double-quantitative rough set models to decision analysis [8,9]. To derive effective rules from data, the novel rough set models proposed by Guo et al.

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Table 1
The review and comparison of the proposed methods in IFIS.

Year	Authors	Researches	Methods
2002	Greco et al.	Rough approximation by dominance relations [20]	DRSA
2007	Xu et al.	IFPR and their application in group decision making [25]	IFPR
2013	Xu et al.	Intuitionistic fuzzy ordered information system [24]	IFOIS
2013	Bing et al.	Dominance-based rough set model in IFIS [21]	IF-DRSA
2014	Huang et al.	Intuitionistic fuzzy multigranulation rough sets [18]	MGRS
2017	Zhang et al.	Generalized dominance rough set models for the dominance IFIS [22]	GDRSA
2019	Huang et al.	DRSA in multi-scale intuitionistic fuzzy decision tables [23]	MSRS
2022	Zhang et al.	A novel rough set method based on different-perspective in IFOIS	SPRS

Table 2
The selection for cadres.

	A_1			A_2		
	Support	Negative	Abstain	Support	Negative	Abstain
X_1	15	3	7	18	5	2
X_2	18	5	2	13	3	9
X_3	13	5	7	10	0	15
X_4	13	10	2	13	10	2
X_5	18	5	2	15	5	5
X_6	20	5	0	20	3	2

maintain a particular position in relation to rule extraction [10,11]. Dai et al. presented a variety of feature selection [12,13] methods based on fuzzy rough sets to reduce dimensionality and remove redundant features. Overall, research based on RST is becoming an academic hot spot.

In 1986, Atanassov proposed the intuitionistic fuzzy set (IFS) [14] to characterize the vagueness of objects accurately. Unlike a traditional fuzzy set, an IFS considers the membership degree, non-membership degree, and hesitation degree when processing fuzzy information, and contains more complete and comprehensive information than traditional fuzzy sets when processing fuzziness and uncertainty. An IFS significantly enhances the description of the characteristics of objects, making the characterization of the objects more accurate and specific. Therefore, the combination of IFSs and RST has become another research hot spot [15,16]. Many researchers have presented important studies in this area. Lu and Lei [17] designed an attribute-reduction algorithm based on intuitionistic fuzzy rough sets (IFRS). Huang et al. [18] integrated IFRS with the concept of multi-granulation to form multi-granulation IFRS models. Additionally, in real-world applications, numerous information tables based on intuitionistic fuzzy environments are referred to as intuitionistic fuzzy information systems (IFIS) [19].

There is a binary relation called the equivalence relation in classical intuitionistic fuzzy information tables, but many problems do not satisfy the equivalence relation in practical applications. To overcome this limitation, Greco et al. presented the dominance-based rough set approach (DRSA) [20]. The DRSA replaces binary relations with dominance relations in information tables. The resulting intuitionistic fuzzy information tables are known as intuitionistic fuzzy ordered information systems (IFOIS). As shown in Table 1, since the DRSA was first proposed, scholars have performed numerous studies on the combination of IFS and DRSA. Bing et al. proposed a dominance-based rough set model for IFIS [21]. Zhang et al. extended the dominance-based rough set model to generalized dominance-based rough set models for IFIS [22]. Huang et al. used DRSA to research multi-scale intuitionistic fuzzy decision tables [23].

The ranking problem of IFS is involved in the process of combining IFS with DRSA. Xu et al. considered dominance relations from the perspective of comparing membership degrees and non-membership degrees, but ignored the generalization of data ranking [24]. Subsequently, Xu et al. proposed the construction of a scoring function and accuracy function to obtain intuitionistic fuzzy partial order relations [25]. In this paper, we propose a novel dominance relation based on the triangular norm and uncertainty metrics to reveal the different focuses of intuitionistic fuzzy data ranking. We were motivated by the fact that the membership and non-membership of IFS are not equally important in most practical problems or considered comprehensively. Additionally, existing sorting methods fail to represent the potential preferences of intuitionistic fuzzy data. Consider the selection of cadres as an example. In Table 2, it is assumed that there are six candidates $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ and 25 voters involved in the selection process. The evaluation index A_1 represents ability and A_2 denotes morality. Each of the 25 voters can support or oppose a candidate, or abstain. For the indicator of ability A_1, if a person exhibits a moderately high or extremely high level of ability, then they will likely receive favorable votes. In contrast, if a person only demonstrates a moderate level of ability, then they may receive opposing votes. However, it is unreasonable to completely deny the ability of such an individual. As a result, in most cases, experts focus on the number of votes in favor of candidates considering the ability index A_1 during the selection process, where a greater number of supporting votes indicates a better candidate. In contrast, regarding the indicator of morality A_2, when a person has good or excellent moral character, they will likely receive supporting votes from everyone. When a person's moral performance is average or they have made no significant mistakes, most people will give them support votes. However, if a person has bad morals or

Table 3
The intuitionistic fuzzy information table.

U	X_1	X_2	X_3	X_4	X_5	X_6
A_1	(0.6,0.1)	(0.7,0.2)	(0.5,0.2)	(0.5,0.4)	(0.7,0.2)	(0.8,0.2)
A_2	(0.7,0.2)	(0.5,0.1)	(0.4,0.0)	(0.5,0.4)	(0.6,0.2)	(0.8,0.1)

commits major mistakes, then they will almost certainly receive opposing votes. Therefore, in most cases, experts are likely to care more about the number of votes against candidates based on the moral indicator A_2. Additionally, we know that the purpose of voting is to allow everyone to express their opinions and views, but there may be some scenarios in which many people abstain from voting and choose a neutral position, which defeats the purpose of voting. The number of abstentions is also a factor that experts must consider. Limiting the number of abstentions will ensure that a vote is meaningful. However, existing sorting methods cannot reflect different focuses on supporting votes, opposing votes, or abstentions.

The essence of this example can be transformed into the IFIS in Table 3. Supporting and opposing votes respectively represent for the degree of membership and non-membership in the IFS, and abstentions represent for the degree of hesitation in the IFS. Representing different concerns in the IFS using a novel sorting method is one of the main motivations of this study. Based on the discussion above, this paper defines strong, weak, and hesitant dominance relations according to the triangular norms, which respectively focus on the consideration of the membership degree, non-membership degree, and hesitation degree.

Different sorting methods correspond to different dominance relations, so each feature can be used to determine an optimal dominance relation [26] according to the connections between the ranking of objects and decision results. Therefore, evaluating the quality of each feature object ranking is an inevitable problem. It is well known that uncertainty measures play a vital role in rough set theory for assessing the importance of attributes and quantifying the inconsistency of data. The incorporation of information entropy [27] has significantly enhanced the development of uncertainty measurement and has been applied in various fields, such as information fusion [28], feature selection [29,30], and data classification [31,32]. Hu et al. investigated ranking mutual entropy and ranking conditional entropy in ordinal classification [33]. These indicators reflect the consistency degree of sample ranking under different features and decisions, meaning they serve as indexes for evaluating the monotonic consistency of ordinal classification. Inspired by this concept, we use the information entropy of different features to evaluate which sorting method is more suitable for a given feature.

Based on the methods described above, we propose adjustable-perspective dominance relations and construct a corresponding IFRS model. Considering the different connections between diverse features and decisions, the optimal dominance relations selected under different feature sets are distinct. Some features represent the attribute sets of dominance relations that focus on membership degree, some features represent the attribute sets of dominance relations that revolve around the non-membership degree, and others represent the attribute sets of dominances relation that focus on the hesitation degree. Based on these three attribute sets representing different perspectives, we investigated the different-perspective rough sets (DPRS) approach in IFOIS while considering both loose and strict conditions. Based on the MGRS [34] concept proposed by Qian, the attribute set of each perspective can be considered as one granulation. Therefore, DPRS is a method for exploring information from a single perspective or multiple perspectives while emphasizing the analysis of problems from dissimilar angles. Additionally, for the attribute sets corresponding to each perspective, we can further incorporate the concept of multi-granulation to mine data information from different emphases and multiple levels to consistently obtain more comprehensive and reasonable problem solutions.

Inspired by the concepts described above, this paper defines three dominance relations based on the triangular norm and explores the adjustable-perspective dominance relation presented in Fig. 1. Based on this adjustable-perspective dominance relation, we design a new model for the single-perspective rough set (SPRS) in IFOIS. Additionally, the attribute set of IFOIS can be divided into the strong dominance relation attribute set A_1, weak dominance relation attribute set A_2, and hesitant dominance relation attribute set A_3 based on adjustable-perspective dominance relations. Subsequently, we present two types of DPRS based on the three dominance relation sets and study their related properties. Additionally, we present a series of experiments to illustrate the superiority, applicability, and scalability of the proposed models. The main contributions of this paper can be summarized as follows.

- A novel relation called the adjustable-perspective dominance relation is constructed, which both retains the reduction effect of unreasonable information in intuitionistic fuzzy data and the evaluation impact of features from different perspectives.
- Based on the adjustable-perspective dominance relation, we established a novel SPRS model. This model not only represents the preference relations between objects, but also reflects the emphasis of data ranking under different features. It follows that the strategy of this model is more in line with the needs of practical applications.
- Regarding the adjustable-perspective dominance relation, the attribute set can be divided into three parts with different emphases. As a result, we define the loose different-perspective rough set model (LDPRS) and strict different-perspective rough set model (SDPRS) according to variant restrictions.
- To investigate the two types of DPRS models further, we explore related properties and rule induction. Additionally, several algorithms for computing the ordinal classification consistency, roughness, and dependence degree of the proposed models are designed.

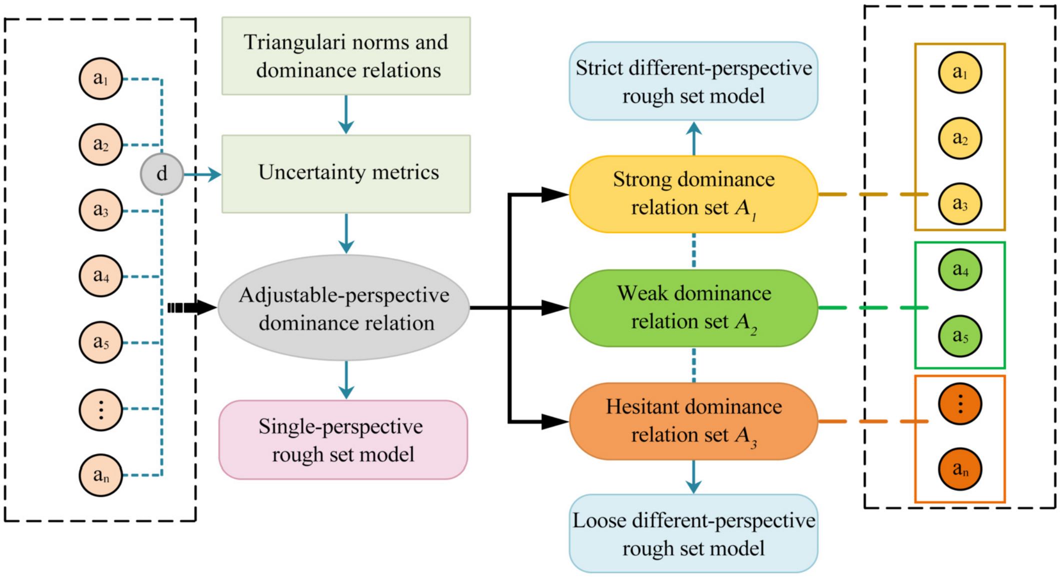


Fig. 1. The motivations of proposed approaches.

- The experimental results demonstrate that the adjustable-perspective dominance relation improves the ordinal classification accuracy, approximate accuracy, and approximate quality of the dominance-based rough set model compared to other dominance relations. Additionally, SPRS improves the rough set model from three perspectives, namely rationality, applicability, and accuracy, which is verified on eight public datasets.

The remainder of this paper is organized as follows. To aid in comprehension, the main motivations and contributions of this study are illustrated in Fig. 1. In Section 2, some basic concepts related to DRSA and triangular norms are reviewed got IFOIS. Section 3 defines the adjustable-perspective dominance relation and establishes the SPRS model. Additionally, this section presents two types of DPRS in IFOIS and further explores the corresponding theorems. In Section 4, a series of experiments are presented based on 8 UCI datasets to verify the effectiveness of the proposed method. Section 5 summarizes and concludes the paper.

2. Preliminaries

In this section, we review some basic concepts related to IFOIS and approaches to comparisons between intuitionistic fuzzy elements.

2.1. IFOIS

Compared to fuzzy sets, the IFS proposed by Atanassov [14] emphasizes the concepts of the non-membership degree and hesitation degree to improve the accuracy of description of the objective world. As a more rigorous structure, an IFS reflects the degree of recognition and disapproval of experts when evaluating objects, which is more consistent with human cognition in terms of solving practical problems.

Let U be a non-empty finite universe set. Then, an IFS on U can be defined as

$$A = \{< x, \mu_A(x), \nu_A(X) > | X \in U\},$$

where the functions $\mu_A : U \to [0,1]$, $\nu_A : U \to [0,1]$ represent the membership degree and non-membership degree of x in A, respectively, and comply with $0 \le \mu_A(x) + \nu_A(x) \le 1$. Furthermore, $\omega_A(x) = 1 - \mu_A(x) - \nu_A(x)$ corresponds to the hesitation degree of x belonging to A. The score function of A is $s(x) = \mu_A(x) - \nu_A(x)$ and its accuracy function is $h(x) = \mu_A(x) + \nu_A(x)$.

Because an IFS can reveal the ambiguity and uncertainty of a dataset more intuitively, some scholars have established IFIS by introducing an IFS into classical information systems. An IFIS can be defined by the following quadruple:

$$\widetilde{I} = (U, AT, V = \{V_{a_j} | a_j \in AT\}, f = \{f_{a_j} | a_j \in AT\}),$$

where $U = \{x_1, x_2, ..., x_m\}$ represents a non-empty, finite universe set, $AT = (a_1, a_2, ..., a_n)$ is a set of conditional attributes, V_{a_j} is a set of non-empty intuitionistic fuzzy numbers with attributes $a_j \in AT$, which is also called the domain of attribute a_j, and $f_{a_j} : U \to V_{a_j}$ represents a function that maps an object in U to an accurate intuitionistic fuzzy number in V_{a_j}. Specifically, $f_a(x) = (\mu_a(x), \nu_a(x))$ for any $x \in U$, $a \in AT$.

In real-world applications, there is typically an increasing partial order and decreasing partial order in an IFIS. If the domain of a conditional attribute is ordered by a decreasing or increasing preference, then that attribute can be considered as a criterion. If all attributes are criteria, then an IFIS is an IFOIS. For simplicity, an IFOIS can be denoted as $\widetilde{I}^{\geq} = (U, AT, V, f)$ [24].

2.2. Ranking problems in IFOIS

For an IFOIS, research on representing the preference relations between objects is the key concept for processing an IFS. Many scholars have proposed methods for comparing two intuitionistic numbers. In this subsection, we will review comparison methods for IFS. Additionally, we highlight the ranking problems in existing approaches.

Definition 2.1. [24] Let $\widetilde{I}^{\geq} = (U, AT, V, f)$ be an IFOIS, where $\forall x, y \in U$, $A \subseteq AT$. The dominance relation $\widehat{R}^{\geq}$ on A can be defined as

$$\widehat{R}_A^{\geq} = \{(x, y) \in U \times U \mid \mu_a(x) \leq \mu_a(y), \nu_a(x) \geq \nu_a(y), \forall a \in A\},$$

where $\widehat{R}_A^{\geq}$ satisfies the reflective, transitive, and asymmetry conditions. $\widehat{[x]}_A^{\geq} = \{y \in U \mid (x, y) \in \widehat{R}_A^{\geq}\}$ denotes the object sets that dominate x with respect to A.

We found that many previous studies are limited because they use this dominance relation to rank objects, meaning conditions are too strict for comparing numerous objects. For example, objects such as (0.5,0.2), (0.4,0.0), and (0.7,0.3) are incomparable when considering the dominance relation $\widehat{R}_A^{\geq}$.

To overcome this limitation, Xu et al. proposed an effective method for measuring the preference relations between objects using a scoring function and accuracy function based on IFS theory [25]. This method has improved applicability compared to the original dominance relation because all objects can be compared. Next, we will review the concept of general dominance relations based on scoring functions and accuracy functions. Suppose the intuitionistic fuzzy number of an object x under the attribute $a \in AT$ can be expressed as $x_a = f(x, a) = (\mu_a(x), \nu_a(x))$ and that its scoring function and accuracy function are $s(x) = \mu_a(x) - \nu_a(x)$ and $h(x) = \mu_a(x) + \nu_a(x)$, respectively. Then, the following definition for comparing two intuitionistic fuzzy numbers can be obtained.

Definition 2.2. [25] Let $f(x, a) = (\mu_a(x), \nu_a(x))$ and $f(y, a) = (\mu_a(y), \nu_a(y))$ be two intuitionistic fuzzy numbers. If $s(x) < s(y)$ holds, then $x < y$. If both $s(x) = s(y)$ and $h(x) < h(y)$ hold, then $x < y$. If $h(x) = h(y)$, then we have $x = y$.

By applying Definition 2.2 to an IFOIS, we can derive the following general dominance relation for IFOIS.

Definition 2.3. [25] Let $\widetilde{I}^{\geq} = (U, AT, V, f)$ be an IFOIS, where $\forall x, y \in U$, $A \subseteq AT$. The general dominance relation $\widetilde{R}^{\geq}$ on A can be defined as

$$\widetilde{R}_A^{\geq} = \{(x, y) \in U \times U \mid f(x, a) \leq f(y, a), \forall a \in A\},$$

where $f(x, a)$ and $f(y, a)$ respectively represent the intuitionistic fuzzy numbers of two objects. Additionally, we derive that the dominance class $\widetilde{[x]}_A^{\geq} = \{y \in U \mid (x, y) \in \widetilde{R}_A^{\geq}\}$ is comprised of objects dominating x. $\widetilde{R}_A^{\geq}$ still satisfies the reflective, transitive, and asymmetry conditions. Additionally, the dominance relation under the decision attribute d can be expressed as $\widetilde{R}_d^{\geq} = \{(x, y) \in U \times U \mid f(x, d) \leq f(y, d)\}$.

Based on the dominance relation and corresponding dominance class described above, we will present a dominance rough set model for IFOIS.

Definition 2.4. Let $\widetilde{I}^{\geq} = (U, AT, V, f)$ be an IFOIS, where for $X \subseteq U$, $A \subseteq AT$. The lower and upper approximations of X with respect to the dominance relation $\widetilde{R}_A^{\geq}$ are defined as follows:

$$\underline{\widetilde{R}_A^{\geq}}(X) = \left\{ x \in U \,\middle|\, \widetilde{[x]}_A^{\geq} \subseteq X \right\},$$
$$\overline{\widetilde{R}_A^{\geq}}(X) = \left\{ x \in U \,\middle|\, \widetilde{[x]}_A^{\geq} \cap X \neq \emptyset \right\}.$$

$\underline{\widetilde{R}_A^{\geq}}(X)$ and $\overline{\widetilde{R}_A^{\geq}}(X)$ are a pair of approximation operators. If $\underline{\widetilde{R}_A^{\geq}}(X) = \overline{\widetilde{R}_A^{\geq}}(X)$, then X is a definable set. Otherwise, it is a rough set. The three regions of X are denoted as $POS(X) = \underline{\widetilde{R}_A^{\geq}}(X)$, $NEG(X) = \sim \underline{\widetilde{R}_A^{\geq}}(X)$, and $BND(X) = \overline{\widetilde{R}_A^{\geq}}(X) - \underline{\widetilde{R}_A^{\geq}}(X)$.

Definition 2.5. Let $\widetilde{I}^{\geq} = (U, AT, V, f)$ be an IFOIS, where for $\forall X \subseteq U$, $A \subseteq AT$. The roughness of X under the dominance relation $\widetilde{R}_A^{\geq}$ is defined as

$$\rho_A^{\geq}(X) = 1 - \frac{\left|\underline{\widetilde{R}_A^{\geq}}(X)\right|}{\left|\overline{\widetilde{R}_A^{\geq}}(X)\right|}.$$

Table 4
Comprehensive assessment of students.

U	a_1	a_2	a_3	d
x_1	(0.8,0.1)	(0.3,0.0)	(0.7,0.2)	A
x_2	(0.4,0.2)	(0.3,0.5)	(0.5,0.0)	C
x_3	(0.5,0.3)	(0.7,0.2)	(0.6,0.2)	B
x_4	(0.5,0.4)	(0.6,0.3)	(0.4,0.3)	B
x_5	(0.6,0.4)	(0.7,0.0)	(0.9,0.1)	A
x_6	(0.4,0.1)	(0.3,0.4)	(0.2,0.4)	C

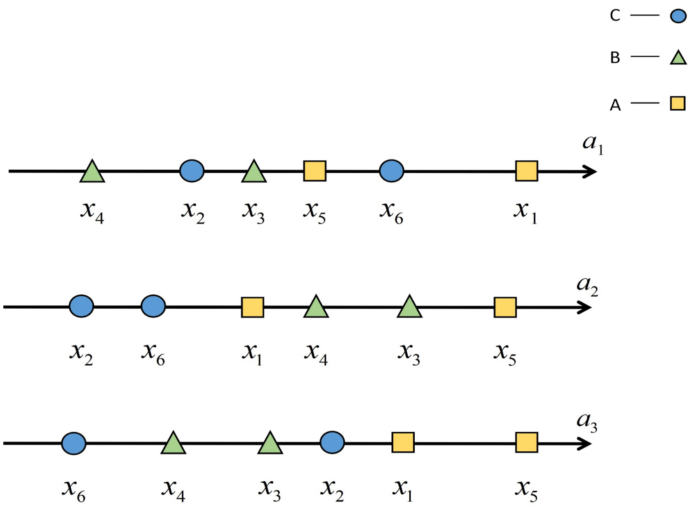


Fig. 2. The ranking of students under attributes a_1 and a_2.

The roughness $\rho_A^{\geq}(X)$ is adopted to express the degree of incomplete knowledge of X. When $\overline{\widetilde{R}_A^{\geq}}(X)=\emptyset$, it is accepted that $\rho_A^{\geq}(X)=1$. Clearly, the accuracy of X can be denoted as $\alpha_A^{\geq}(X)=1-\rho_A^{\geq}(X)$.

For an intuitionistic fuzzy ordered decision information system (IFODIS), the decision attribute d determines the partition of U, which is denoted as $U/d=\{D_i | i\in\{1,2,...,N\}(N\leq|U|)\}$, where D_i is an equivalence class. Then, there is a preference relationship between each decision class D_i, $D_1\prec\cdots\cdots\prec D_j\cdots\cdots\prec D_N$. The upward and downward unions of the decision classes can be denoted as $D_i^+=\cup D_j(j\geq i)$ and $D_i^-=\cup D_j(j\leq i)$, respectively The quality of approximation, which is also called the degree of dependence, plays an important role in reflecting the consistency degree of object rankings in terms of conditional attributes and decision attributes. Next, we introduce the concepts of dependency.

Definition 2.6. Given an IFODIS $\widetilde{I}^{\geq}=(U,AT\cup d,V,f)$, $A\subseteq AT$, the dependence degree of D is defined as

$$\gamma_A(D^+)=\frac{\sum\limits_{i=1}^{|N|}\left|\underline{\widetilde{R}_A^{\geq}}(D_i^+)\right|}{\sum\limits_{i=1}^{|N|}\left|D_i^+\right|},$$

where D_i^+ represents classes that dominate D_i and $|*|$ denotes the cardinality of set $*$.

One can see that a dependence degree based on the general dominance relation cannot reflect the consistency of object rankings in an IFOIS. The following example illustrates this limitation.

Example 2.1. Table 4 summarizes the comprehensive achievements of students. There are six students $U=\{x_1,x_2,x_3,x_4,x_5,x_6\}$ and three evaluation criteria $AT=\{a_1,a_2,a_3\}$ in this evaluation. a_1 represents the performance of students, a_2 represents the psychological health of students, and a_3 represents participation in extracurricular activities. The decision attribute d represents the total level $(C\prec B\prec A)$ of students in this evaluation.

In this comprehensive evaluation, it should be noted that experts are more inclined toward the membership degree of student performance and non-membership degree of student psychological health, and focus on the hesitation degree of student extracurricular activities. According to the general dominance relation, we can obtain a ranking of students under attributes a_1,a_2,a_3 and map the results onto coordinate axes, as shown in Fig. 2, where ◯, Δ, and □ represent objects in classes C, B, and A, respectively.

Fig. 2 clearly reveals that the ranking of students under attributes a_1 and d has a high degree of inconsistency. x_2 and x_6 in grade C are assigned to a relatively high level, whereas x_4 in grade B and x_5 in grade A are assigned to a relatively low level. The degree of consistency under attribute a_1 is $\gamma_{a_1}(D^+) = 0.67$. There are also inconsistencies in the ranking of students between attributes a_2 and d. It is clear that x_1 in grade A is assigned a relatively low level and the degree of consistency under attribute a_2 is $\gamma_{a_2}(D^+) = 0.92$. Furthermore, in the ranking of students under a_3, x_2 is assigned a relatively high level. The degree of consistency under attribute a_3 is $\gamma_{a_3}(D^+) = 0.83$. These sorting results clearly do not conform to the expert judgment logic.

Analysis revealed that the ranking of objects under the general dominance relation cannot express the characteristics of experts paying more attention to the membership degree, non-membership degree, or hesitation degree under a certain attribute. This is because the general dominance relation fails to judge and reflect the emphases of attributes. Therefore, the general dominance relation cannot explore the connotations of data deeply and comprehensively. In real-world applications, there are many intuitionistic fuzzy cases in which the degree of membership and non-membership are not equally important. Such problems require the dominance relation to describe the order relations of objects from different perspectives to mimic the logic of human cognition.

To overcome this limitation, we introduce triangular norms and conorms (also called t-norms and t-conorms). As important concepts in set theory, triangular norms and conorms play a vital role in the comparison of fuzzy sets and other set fields. We now recall some basic concepts of triangle norms.

- Let N be a mapping $N : [0, 1] \times [0, 1] \to [0, 1]$ for $\forall \mu, \nu \in [0, 1]$. If N satisfies the following conditions:

(1) Boundary conditions: $N(0) = 1, N(1) = 0$,
(2) Monotonicity: If $\mu \le \nu$, then $N(\mu) \ge N(\nu)$,

then the mapping N is called a fuzzy complement mapping. If there is a continuous and strictly decreasing complement, then the fuzzy negation is strictly a fuzzy complement. Additionally, the fuzzy complement mapping N_s is called a standard fuzzy complement operator if and only if $N_s(\mu) = 1 - \mu$ holds for $\forall \mu \in [0, 1]$.

- Let T be a mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ for $\forall \mu, \nu, t \in [0, 1]$. If T satisfies the following conditions:

(1) Commutativity: $T(\mu, \nu) = T(\nu, \mu)$,
(2) Associativity: $T(\mu, T(\nu, t)) = T(T(\mu, \nu), t)$,
(3) Boundary condition: $\forall \mu \in [0, 1]$, $T(\mu, 1) = \mu$,
(4) Monotonicity: If $\mu \le \nu$, then $T(t, \mu) \le T(t, \nu)$,
then the mapping T is called a triangular norm or t-norm for short.

- Let S be a mapping $S : [0, 1] \times [0, 1] \to [0, 1]$ for $\forall \mu, \nu, t \in [0, 1]$. If S satisfies the following conditions:

(1) Commutativity: $S(\mu, \nu) = S(\nu, \mu)$,
(2) Associativity: $S(\mu, S(\nu, t)) = S(S(\mu, \nu), t)$,
(3) Boundary condition: $\forall x \in [0, 1]$, $S(x, 1) = x$,
(4) Monotonicity: If $\mu \le \nu$, then $S(t, \mu) \le S(t, \nu)$,
then the mapping S is called a fuzzy union or t-conorm for short.

Considering the above definitions, it can be obtained that the triangular norm (t-norm) and fuzzy union (t-conorm) are both in effect with respect to the fuzzy complement operator N when $N(T(\mu, \nu)) = S(N(\mu), N(\nu))$ or $N(S(\mu, \nu)) = T(N(\mu), N(\nu))$ hold for $\forall \mu, \nu \in [0, 1]$. Let (T, S, N) be a dual triple, where the t-norm and t-conorm are in effect with regard to N. Then, the min-max dual triple function can be defined as

$$Min - Max : F(x) = (min\,(\mu_x, \nu_x)\,, max\,(\mu_x, \nu_x)\,, N)\,,$$

where $T(\mu, \nu) = min\,(\mu_x, \nu_x)$, $S(\mu, \nu) = max\,(\mu_x, \nu_x)$. It should be noted that $N(T(\mu, \nu)) = 1 - min\,(\mu_x, \nu_x) = max\,(1 - \mu_x, 1 - \nu_x) = S(N(\mu), N(\nu))$ and $N(S(\mu, \nu)) = 1 - max\,(\mu_x, \nu_x) = min\,(1 - \mu_x, 1 - \nu_x) = T(N(\mu), N(\nu))$ if and only if the operators T and S are both in effect.

In accordance with the min-max dual triple, three different types of dominance relations can be derived.

3. SPRS and DPRS in IFOIS

This section will focus on a novel relation called the adjustable-perspective dominance relation based on the triangular norms. We investigate the SPRS and DPRS based on this relation.

3.1. SPRS in IFOIS

We wish to overcome the limitations of object ranking based on a simple scoring function and accuracy function. Therefore, we utilize the triangular norm operator to establish the adjustable-perspective dominance relation, which more precisely reflects the different emphases of object ranking in practical problems.

Definition 3.1. Let $\widetilde{I}^{\geq} = (U, AT, V, f)$ be an IFOIS $\forall x, y \in U$, $A \subseteq AT$. In combination with the triangular norm operators in the min-max dual triple, the dominance relation $\widetilde{R}^{G\geq}$ under attribute A can be defined as

$$\widetilde{R}_A^{G\geq} = \{ (x, y) \in U \times U \mid G(x, a) \leq G(y, a), \forall a \in A\},$$

where $G(x,a) \in \{T(x,a), S(x,a), H(x,a)\}$, and $T(x,a) = T(\mu_a(x), N(\nu_a(x))) = min(\mu_a(x), 1-\nu_a(x))$, $S(x,a) = S(\mu_a(x), N(\nu_a(x))) = max(\mu_a(x), 1-\nu_a(x))$, $H(x,a) = (T(\mu_a(x), N(\nu_a(x)) - S(\mu_a(x), N(\nu_a(x))) = (min(\mu_a(x), 1-\nu_a(x)) - max(\mu_a(x), 1-\nu_a(x)))$. We respectively call $\widetilde{R}_A^{T\geq}$, $\widetilde{R}_A^{S\geq}$, and $\widetilde{R}_A^{H\geq}$ the strong dominance relation, weak dominance relation and hesitant dominance relation when $G(x,a)$ is taken as $T(x,a)$, $S(x,a)$, $H(x,a)$. Here, $\widetilde{R}_A^{T\geq}$ indicates that attribute A focuses on the membership degree in the object ranking. $\widetilde{R}_A^{S\geq}$ indicates that A emphasizes non-membership during object ranking. $\widetilde{R}_A^{H\geq}$ indicates that A focuses on the hesitation degree.

The corresponding dominance classes can be defined as

$$\widetilde{[x]}_A^{G\geq} = \{ y \in U \mid (x, y) \in \widetilde{R}_A^{G\geq}.$$

Similarly, $\widetilde{[x]}_A^{G\geq}$ refers to $\widetilde{[x]}_A^{T\geq}$, $\widetilde{[x]}_A^{S\geq}$, and $\widetilde{[x]}_A^{H\geq}$ when $G(x,a)$ is taken as $T(x,a)$, $S(x,a)$, and $H(x,a)$, respectively. Clearly, $\widetilde{[x]}_A^{G\geq}$ satisfies the reflective, transitive, and asymmetry conditions.

During the ranking process of intuitionistic fuzzy data, existing dominance relations are often evaluated from a general and unified level, meaning only the difference and sum values between the membership degree and non-membership degree are considered. In reality, based on the complex and biased evaluations of humans, membership and non-membership are not typically equally important. Accordingly, a general dominance relation alone is not sufficient to summarize and synthesize the ranking considerations of humans. Therefore, we aim to establish three types of dominance relations $\widetilde{R}_A^{T\geq}$, $\widetilde{R}_A^{S\geq}$, and $\widetilde{R}_A^{H\geq}$ to represent the process of object ranking from different perspectives, which makes the ranking results of intuitionistic fuzzy data more intelligent and comprehensive.

Example 3.1. There are various street interviews every year. Assuming that there are six candidates on a street interview, an expert's assessment with six people answering mathematics questions can be expressed as (0.7,0.1), (0.3,0.2), (0.4,0.3), (0.6,0.4), (0.4,0.5), (0.5,0.1). Conforming to aforementioned the dominance relation of Definition 3.1. The dominance relations of these six candidates can be computed as follows:

$$\begin{array}{lll}
T(x_1,a) = 0.7 & S(x_1,a) = 0.9 & H(x_1,a) = -0.2 \\
T(x_2,a) = 0.3 & S(x_2,a) = 0.8 & H(x_2,a) = -0.5 \\
T(x_3,a) = 0.4 & S(x_3,a) = 0.7 & H(x_3,a) = -0.3 \\
T(x_4,a) = 0.6 & S(x_4,a) = 0.6 & H(x_4,a) = 0.0 \\
T(x_5,a) = 0.4 & S(x_5,a) = 0.5 & H(x_5,a) = -0.1 \\
T(x_6,a) = 0.5 & S(x_6,a) = 0.9 & H(x_6,a) = -0.4 \\
T: x_2 \leq x_3 = x_5 \leq x_6 \leq x_4 \leq x_1; & S: x_5 \leq x_4 \leq x_3 \leq x_2 \leq x_6 = x_1; & H: x_2 \leq x_6 \leq x_3 \leq x_1 \leq x_5 \leq x_4.
\end{array}$$

The results of our evaluations reveal that the three dominance relations mentioned above represent different preferences on data. The strong dominance class $\widetilde{[x]}_A^{T\geq}$ focuses on the membership degree of data, the weak dominance class $\widetilde{[x]}_A^{S\geq}$ focuses on the non-membership degree, and the hesitation dominance class $\widetilde{[x]}_A^{H\geq}$ focuses on the hesitation degree. In the Example 3.1, the membership degree of the evaluation results represents the accuracy of answering mathematical questions, the non-membership degree represents the error rate, and the hesitation degree represents the time required to answer questions. Distinctly, all of these factors can be considered as important evaluation indicators for experts to measure object rankings. However, the general dominance relation only focuses on the difference and sum between membership and non-membership while ignoring these important research factors, which leads to insufficient information extraction and inaccurate approximation results. The three dominance relations proposed in this paper are able to extract information from different perspectives and comprehensively evaluate ranking indexes so that ranking results are more in line with actual needs.

Considering the complex and varied features of practical problems, deriving a dominance relation adapted to each feature is essential. Hu et al. [33] proposed using the dominance conditional entropy (DCE) and dominance mutual entropy (DME) to measure the consistency of datasets between features and decisions in an ordered information system. The dominance class induced by the dominance relation is the core element of DCE, and the ranking consistency of the conditional dominance classes and decision dominance classes of different dominance relations vary. Therefore, we combine the three aforementioned dominance relations with DCE and employ multi-dominance conditional entropy (MDCE) to evaluate the consistency of object ranking from multiple perspectives. This not only maintains the values of attributes, but also reduces the impact of unreasonable information contained in data. Additionally, this evaluation process deeply reflects the connections between the order of features and decisions from different perspectives, meaning the sorting results are more applicable and reliable.

Definition 3.2. Given an IFODIS $\widetilde{I}^{\geq} = (U, AT \cup d, V, f)$, where $A \subseteq AT$, the intuitionistic fuzzy MDCE of A relative to d is defined as

$$\widetilde{RC}^{G\geq}_{d|A}(U) = -\frac{1}{|U|}\sum_{j=1}^{m}\log\frac{\left|\widetilde{[x_j]}^{G\geq}_A \cap D^+_d(x_j)\right|}{\left|D^+_d(x_j)\right|},$$

where $\widetilde{[x_j]}^{G\geq}_A$ indicates three dominance classes and $\frac{|\widetilde{[x_j]}^{G\geq}_A \cap D^+_d(x_j)|}{|D^+_d(x_j)|}$ can be considered as the core of $\widetilde{RC}^{G\geq}_{d|A}$. If the value of $\frac{|\widetilde{[x_j]}^{G\geq}_A \cap D^+_d(x_j)|}{|D^+_d(x_j)|}$ is higher when $\widetilde{[x_j]}^{G\geq}_A$ takes the form of one of $\widetilde{[x_j]}^{T\geq}_A$, $\widetilde{[x_j]}^{S\geq}_A$, and $\widetilde{[x_j]}^{H\geq}_A$, then the dominance relations corresponding to $\widetilde{[x_j]}^{T\geq}_A$, $\widetilde{[x_j]}^{S\geq}_A$, or $\widetilde{[x_j]}^{H\geq}_A$ are the relations that make the ranked objects have the highest degree of consistency under the considered features and decisions. Therefore, the greater the value of $\frac{\left|\widetilde{[x_j]}^{G\geq}_A \cap D^+_d(x_j)\right|}{\left|D^+_d(x_j)\right|}$, the greater the significance of the dominance relation.

We now focus on a novel relation called the adjustable-perspective dominance relation.

Definition 3.3. Let $\widetilde{I}^{\geq} = (U, AT \cup d, V, f)$ be an IFODIS $\forall x, y \in U$, where $A \subseteq AT$. The adjustable-perspective dominance relation $\widetilde{R}^{G_S\geq}$ under attribute A can be defined as

$$\widetilde{R}^{G_S\geq}_A = \underset{G\in\{T,S,H\}}{\arg\min}\ \widetilde{RC}^{G\geq}_{d|A}(U)$$

and the adjustable-perspective dominance class induced by this relation is defined as

$$\widetilde{[x]}^{G_S\geq}_A = \{y \in U \mid (x, y) \in \widetilde{R}^{G_S\geq}_A\}.$$

Clearly, $\widetilde{[x]}^{G_S\geq}_A$ is different from $\widetilde{[x]}^{G\geq}_A$ for $\widetilde{[x]}^{G_S\geq}_A$. $\widetilde{[x]}^{G_S\geq}_A$ is comprised of all objects dominating x according to the adjustable-perspective dominance relation.

Proposition 3.1. *Let $\widetilde{I}^{\geq} = (U, AT \cup \{d\}, V, f)$ be an IFODIS $U = \{x_1, x_2, ..., x_n\}$, where $B, A \subseteq AT$. Then, we have the following properties:*

(1) *If $B \subseteq AT$, then $\widetilde{R}^{G_S\geq}_A \subseteq \widetilde{R}^{G_S\geq}_B$, $\widetilde{[x]}^{G_S\geq}_A \subseteq \widetilde{[x]}^{G_S\geq}_B$,*

(2) $x_t \in \widetilde{[x_s]}^{G_S\geq}_A \Leftrightarrow \widetilde{[x_t]}^{G_S\geq}_A \subseteq \widetilde{[x_s]}^{G_S\geq}_A$, $\widetilde{[x_s]}^{G_S\geq}_A = \cup\{\widetilde{[x_t]}^{G_S\geq}_A \,\Big|\, x_t \in \widetilde{[x_s]}^{G_S\geq}_A\}$,

(3) *if $\mu(x) + \nu(x) = 1$, then $\widetilde{[x]}^{G_S\geq}_A = \widetilde{[x]}^{T\geq}_A = \widetilde{[x]}^{S\geq}_A$.*

Proof. (1) $\sim$ (2) The proof of these properties can be obtained based on Definitions 3.1 and 3.3.
(3) If $\mu(x) + \nu(x) = 1$, $T(x,a) = T(\mu_a(x), N(\nu_a(x))) = min(\mu_a(x), 1 - \nu_a(x)) = min(\mu_a(x), \mu_a(x)) = \mu_a(x)$, $S(x,a) = S(\mu_a(x), N(\nu_a(x))) = max(\mu_a(x), 1 - \nu_a(x)) = max(\mu_a(x), \mu_a(x)) = \mu_a(x)$, then $T(x,a) = S(x,a)$ holds. Therefore, we can obtain $\widetilde{[x]}^{G_S\geq}_A = \widetilde{[x]}^{T\geq}_A = \widetilde{[x]}^{S\geq}_A$.

Definition 3.4. Let $\widetilde{I}^{\geq} = (U, AT, V, f)$ be an IFOIS for $X \subseteq U$, where $A \subseteq AT$. The lower and upper approximations of X with respected to the adjustable-perspective dominance relation $\widetilde{R}^{G_S\geq}_A$ are defined as

$$\underline{\widetilde{R}^{G_S\geq}_A}(X) = \left\{x \in U \,\middle|\, \widetilde{[x]}^{G_S\geq}_A \subseteq X)\right\},$$
$$\overline{\widetilde{R}^{G_S\geq}_A}(X) = \left\{x \in U \,\middle|\, \widetilde{[x]}^{G_S\geq}_A \cap X \neq \emptyset)\right\}.$$

The pair $(\underline{\widetilde{R}^{G_S\geq}_A}(X), \overline{\widetilde{R}^{G_S\geq}_A}(X))$ is called an SPRS. Three regions of X are defined as $POS(X) = \underline{\widetilde{R}^{G_S\geq}_A}(X)$, $NEG(X) = \sim \underline{\widetilde{R}^{G_S\geq}_A}(X)$, and $BND(X) = \overline{\widetilde{R}^{G_S\geq}_A}(X) - \underline{\widetilde{R}^{G_S\geq}_A}(X)$.

Proposition 3.2. *Let $\widetilde{I}^{\geq}=(U, AT, V, f)$ be an IFOIS $X, Y \subseteq U$, where $A \subseteq AT$. Then, we have the following properties:*

(1) $\underline{\widetilde{R}_A^{G_s\geq}}(X) \subseteq X \subseteq \overline{\widetilde{R}_A^{G_s\geq}}(X)$,

(2) $\underline{\widetilde{R}_A^{G_s\geq}}(\sim X) =\sim \overline{\widetilde{R}_A^{G_s\geq}}(X)$, $\overline{\widetilde{R}_A^{G_s\geq}}(\sim X) =\sim \underline{\widetilde{R}_A^{G_s\geq}}(X)$,

(3) $X \subseteq Y \Rightarrow \underline{\widetilde{R}_A^{G_s\geq}}(X) \subseteq \underline{\widetilde{R}_A^{G_s\geq}}(Y)$, $\overline{\widetilde{R}_A^{G_s\geq}}(X) \subseteq \overline{\widetilde{R}_A^{G_s\geq}}(Y)$.

Proof. The properties above can be derived directly from Definition 3.4.

This yields the conclusion that $\widetilde{R}_A^{G_s\geq}$ eventually chooses an optimal dominance relation. This optimal dominance relation can be either the strong dominance relation, weak dominance relation, or hesitant dominance relation. The selection of the optimal dominance relation directly reveals whether the emphasis under this attribute is the membership degree, non-membership degree, or hesitation degree. In contrast to a weighted IFS, the weight coefficients of the membership degree and non-membership degree are assigned by experts before processing such IFS. The reliability of the weights assigned by experts is an issue worthy of discussion. Therefore, the adjustable-perspective dominance relation is crucial for IFIS. It can objectively classify attributes of different focuses through the connection of features and decisions, and automatically adjust to the optimal dominance relation that represents the target attribute. This not only further explores the relations between decisions and features, but also reduces the irrationality and inconsistency of order classification. Below, we introduce an example to demonstrate the adjustable-perspective dominance relation in IFOIS.

Example 3.2. (Continued from Example 2.1) In order to compute the ranking results of students based on the adjustable-perspective dominance relation, we first calculate the dominance class of strong dominance relation, weak dominance relation and hesitant dominance relation.

Three kinds of dominance classes with respect to a_1 are computed.

$\widetilde{[x_1]}_{a_1}^{T\geq}=\{x_1\}$; $\widetilde{[x_1]}_{a_1}^{S\geq}=\{x_1, x_6\}$; $\widetilde{[x_1]}_{a_1}^{H\geq}=\{x_1, x_4, x_5\}$;
$\widetilde{[x_2]}_{a_1}^{T\geq}=\{x_1, x_2, x_3, x_4, x_5, x_6\}$; $\widetilde{[x_2]}_{a_1}^{S\geq}=\{x_1, x_2, x_6\}$; $\widetilde{[x_2]}_{a_1}^{H\geq}=\{x_1, x_2, x_3, x_4, x_5\}$;
$\widetilde{[x_3]}_{a_1}^{T\geq}=\{x_1, x_3, x_4, x_5\}$; $\widetilde{[x_3]}_{a_1}^{S\geq}=\{x_1, x_2, x_3, x_6\}$; $\widetilde{[x_3]}_{a_1}^{H\geq}=\{x_1, x_3, x_4, x_5\}$;
$\widetilde{[x_4]}_{a_1}^{T\geq}=\{x_1, x_3, x_4, x_5\}$; $\widetilde{[x_4]}_{a_1}^{S\geq}=\{x_1, x_2, x_3, x_4, x_5, x_6\}$; $\widetilde{[x_4]}_{a_1}^{H\geq}=\{x_1, x_4, x_5\}$;
$\widetilde{[x_5]}_{a_1}^{T\geq}=\{x_1, x_5\}$; $\widetilde{[x_5]}_{a_1}^{S\geq}=\{x_1, x_2, x_3, x_4, x_5, x_6\}$; $\widetilde{[x_5]}_{a_1}^{H\geq}=\{x_5\}$;
$\widetilde{[x_6]}_{a_1}^{T\geq}=\{x_1, x_2, x_3, x_4, x_5, x_6\}$. $\widetilde{[x_6]}_{a_1}^{S\geq}=\{x_1, x_6\}$. $\widetilde{[x_6]}_{a_1}^{H\geq}=\{x_1, x_2, x_3, x_4, x_5, x_6\}$.

Three kinds of dominance classes with respect to a_2 are computed.

$\widetilde{[x_1]}_{a_2}^{T\geq}=\{x_1, x_2, x_3, x_4, x_5, x_6\}$; $\widetilde{[x_1]}_{a_2}^{S\geq}=\{x_1, x_5\}$; $\widetilde{[x_1]}_{a_2}^{H\geq}=\{x_1, x_2, x_3, x_4, x_5, x_6\}$;
$\widetilde{[x_2]}_{a_2}^{T\geq}=\{x_1, x_2, x_3, x_4, x_5, x_6\}$; $\widetilde{[x_2]}_{a_2}^{S\geq}=\{x_1, x_2, x_3, x_4, x_5, x_6\}$; $\widetilde{[x_2]}_{a_2}^{H\geq}=\{x_2, x_3, x_4\}$;
$\widetilde{[x_3]}_{a_2}^{T\geq}=\{x_3, x_5\}$; $\widetilde{[x_3]}_{a_2}^{S\geq}=\{x_1, x_3, x_5\}$; $\widetilde{[x_3]}_{a_2}^{H\geq}=\{x_3, x_4\}$;
$\widetilde{[x_4]}_{a_2}^{T\geq}=\{x_3, x_4, x_5\}$; $\widetilde{[x_4]}_{a_2}^{S\geq}=\{x_1, x_3, x_4, x_5\}$; $\widetilde{[x_4]}_{a_2}^{H\geq}=\{x_3, x_4\}$;
$\widetilde{[x_5]}_{a_2}^{T\geq}=\{x_3, x_5\}$; $\widetilde{[x_5]}_{a_2}^{S\geq}=\{x_1, x_5\}$; $\widetilde{[x_5]}_{a_2}^{H\geq}=\{x_2, x_3, x_4, x_5, x_6\}$;
$\widetilde{[x_6]}_{a_2}^{T\geq}=\{x_1, x_2, x_3, x_4, x_5, x_6\}$. $\widetilde{[x_6]}_{a_2}^{S\geq}=\{x_1, x_3, x_4, x_5, x_6\}$. $\widetilde{[x_6]}_{a_2}^{H\geq}=\{x_2, x_3, x_4, x_5, x_6\}$.

Three kinds of dominance classes with respect to a_3 are computed.

$\widetilde{[x_1]}_{a_3}^{T\geq}=\{x_1, x_5\}$; $\widetilde{[x_1]}_{a_3}^{S\geq}=\{x_1, x_2, x_3, x_5\}$; $\widetilde{[x_1]}_{a_3}^{H\geq}=\{x_1, x_5\}$;
$\widetilde{[x_2]}_{a_3}^{T\geq}=\{x_1, x_2, x_3, x_5\}$; $\widetilde{[x_2]}_{a_3}^{S\geq}=\{x_2\}$; $\widetilde{[x_2]}_{a_3}^{H\geq}=\{x_1, x_2, x_3, x_4, x_5, x_6\}$;
$\widetilde{[x_3]}_{a_3}^{T\geq}=\{x_1, x_3, x_5\}$; $\widetilde{[x_3]}_{a_3}^{S\geq}=\{x_1, x_2, x_3, x_5\}$; $\widetilde{[x_3]}_{a_3}^{H\geq}=\{x_1, x_3, x_5\}$;
$\widetilde{[x_4]}_{a_3}^{T\geq}=\{x_1, x_2, x_3, x_4, x_5\}$; $\widetilde{[x_4]}_{a_3}^{S\geq}=\{x_1, x_2, x_3, x_4, x_5\}$; $\widetilde{[x_4]}_{a_3}^{H\geq}=\{x_1, x_3, x_4, x_5\}$;
$\widetilde{[x_5]}_{a_3}^{T\geq}=\{x_5\}$; $\widetilde{[x_5]}_{a_3}^{S\geq}=\{x_1, x_5\}$; $\widetilde{[x_5]}_{a_3}^{H\geq}=\{x_5\}$;
$\widetilde{[x_6]}_{a_3}^{T\geq}=\{x_1, x_2, x_3, x_4, x_5, x_6\}$. $\widetilde{[x_6]}_{a_3}^{S\geq}=\{x_1, x_2, x_3, x_4, x_5, x_6\}$. $\widetilde{[x_6]}_{a_3}^{H\geq}=\{x_1, x_3, x_4, x_5, x_6\}$.

The decision dominance class of each object can be obtained.

$$D_1^+(x_1)=D_1^+(x_5)=\{x_1, x_5\}; \quad D_2^+(x_3)=D_2^+(x_4)=\{x_1, x_3, x_4, x_5\}; \quad D_3^+(x_2)=D_3^+(x_6)=\{x_1, x_2, x_3, x_4, x_5, x_6\}.$$

Then, we are able to acquire the dominance conditional entropy of a_1, a_2, a_3 relative to d through computation.

$\widetilde{RC}_{d|a_1}^{H\geq}(U) > \widetilde{RC}_{d|a_1}^{S\geq}(U) > \widetilde{RC}_{d|a_1}^{T\geq}(U)$;
$\widetilde{RC}_{d|a_2}^{H\geq}(U) > \widetilde{RC}_{d|a_2}^{T\geq}(U) > \widetilde{RC}_{d|a_2}^{S\geq}(U)$;
$\widetilde{RC}_{d|a_3}^{S\geq}(U) > \widetilde{RC}_{d|a_3}^{T\geq}(U) > \widetilde{RC}_{d|a_3}^{H\geq}(U)$.

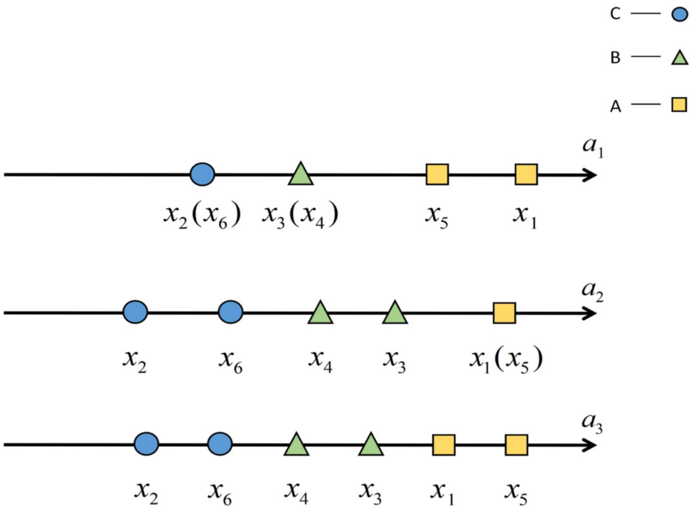


Fig. 3. The revised ranking of students under attributes a_1 and a_2.

Table 5
Dependencies based on proposed relations.

	$\widetilde{R}_A^{T\geq}$		$\widetilde{R}_A^{S\geq}$		$\widetilde{R}_A^{H\geq}$		$\widetilde{R}_A^{G_S\geq}$	
	$\gamma_a(D^+)$	$\widetilde{\gamma}_a(D^+)$	$\gamma_a(D^+)$	$\widetilde{\gamma}_a(D^+)$	$\gamma_a(D^+)$	$\widetilde{\gamma}_a(D^+)$	$\gamma_a(D^+)$	$\widetilde{\gamma}_a(D^+)$
a_1	0.67	1 ↑	0.67	0.50 ↓	0.67	0.92 ↑	0.67	1 ↑
a_2	0.92	0.75 ↓	0.92	1 ↑	0.92	0.67 ↓	0.92	1 ↑
a_3	0.83	0.92 ↑	0.83	0.50 ↓	0.83	1 ↑	0.83	1 ↑

We can directly obtain the adjustable-perspective dominance relation $\widetilde{R}_A^{G_S\geq}$ under attributes a_1, a_2, and a_3 as

$$\widetilde{R}_{a_1}^{G_S\geq}=\widetilde{R}_{a_1}^{T\geq},\ \widetilde{R}_{a_2}^{G_S\geq}=\widetilde{R}_{a_2}^{S\geq},\ \widetilde{R}_{a_3}^{G_S\geq}=\widetilde{R}_{a_3}^{H\geq}.$$

By conforming to relation above, we find that the attribute a_1 prefers the membership degree. a_2 focuses on the non-membership degree and a_3 focuses on the hesitation degree. Additionally, we get the revised ranking of objects presented in Fig. 3 and the dependence degree of D on attributes a_1, a_2, a_3 is presented in Table 5.

In Fig. 3, the object in the brackets is at the same level as the object outside the brackets. Intuitively, Fig. 3 reveals that the revised ranking of the objects on the basis of the adjustable-perspective dominance relation remains consistent with the decision results, which is reasonable and meets the requirements of real-world applications.

Table 5 reveals that the ranking of each attribute under different dominance relations exhibits a different consistency degree with the decision results. Specifically, the strong dominance relation, weak dominance relation, and hesitant dominance relation are more suitable for processing the order relations between objects with respect to attributes a_1, a_2, and a_3, respectively. Therefore, based on the degree of dependence, it can be further determined that attribute a_1 focuses on the membership degree, attribute a_2 focuses on the non-membership degree, and attribute a_3 focuses on the hesitant degree. The adjustable-perspective dominance relation not only reflects the different focus of each attribute, but also adjusts to the optimal dominance relation that adapts to each attribute.

In Example 3.2, only three attributes were discussed for issues with different focuses. However, in real-world applications, there are many problems with numerous attributes and objects, and these attributes have different emphases. Because problems can be analyzed from multiple angles and levels, multi-perspective analysis can consistently obtain more comprehensive and reasonable solutions to problems. Therefore, it is necessary to investigate the DPRS model.

3.2. DPRS in IFOIS

For an intuitionistic fuzzy ordered dataset, there are far more than three attributes. In Example 3.2 above, the strong dominance relation of a_1 indicated that a_1 focuses on the membership degree. The relation of a_2 is the weak dominance relation, meaning a_2 emphasizes the non-membership degree. The relation of a_3 is the hesitant dominance relation, meaning a_3 focuses the hesitation degree. Each attribute subset can determine the corresponding dominance relation based on the adjustable-perspective dominance relation. As shown in Fig. 4, all attributes sets in IFOIS can be divided into three types of dominance relations according to different focuses, namely the attribute sets of the strong dominance relation, weak dominance relation, and hesitant dominance relation.

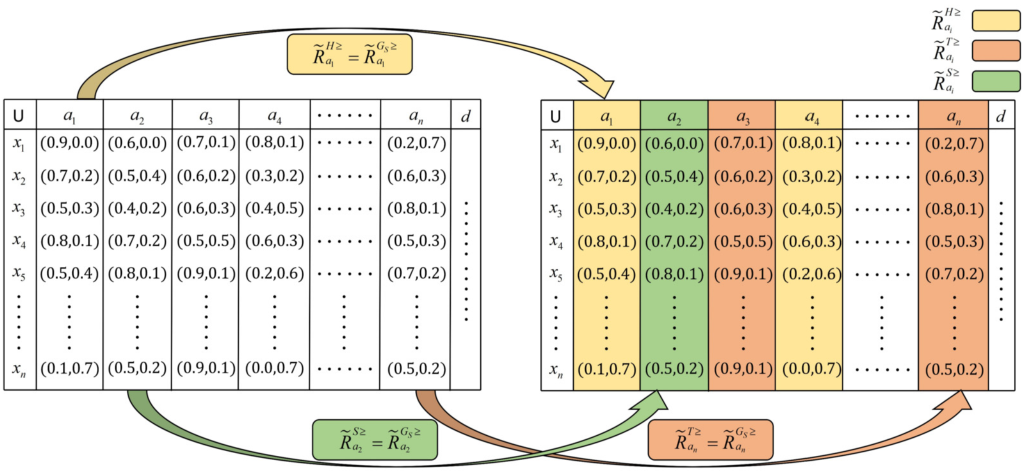


Fig. 4. Three types of dominance relations in IFOIS.

Table 6
Medical evaluation system.

U	a_1	a_2	a_3	a_4	a_5	a_6	d
x_1	(0.9,0.1)	(0.5,0.0)	(0.9,0.0)	(0.6,0.1)	(0.4,0.5)	(0.5,0.4)	A
x_2	(0.4,0.5)	(0.2,0.7)	(0.1,0.6)	(0.3,0.6)	(0.5,0.0)	(0.4,0.1)	C
x_3	(0.5,0.3)	(0.6,0.3)	(0.6,0.2)	(0.5,0.4)	(0.3,0.5)	(0.5,0.3)	B
x_4	(0.5,0.4)	(0.5,0.4)	(0.6,0.3)	(0.6,0.4)	(0.4,0.4)	(0.7,0.1)	B
x_5	(0.7,0.2)	(0.6,0.1)	(0.7,0.1)	(0.5,0.1)	(0.8,0.1)	(0.7,0.2)	A
x_6	(0.4,0.1)	(0.3,0.5)	(0.3,0.3)	(0.2,0.6)	(0.2,0.3)	(0.3,0.0)	C
x_7	(0.6,0.2)	(0.6,0.3)	(0.6,0.4)	(0.8,0.2)	(0.3,0.4)	(0.5,0.3)	B
x_8	(0.5,0.1)	(0.5,0.3)	(0.6,0.3)	(0.4,0.3)	(0.5,0.3)	(0.4,0.4)	B
x_9	(0.2,0.5)	(0.2,0.6)	(0.4,0.5)	(0.1,0.8)	(0.1,0.2)	(0.1,0.3)	C
x_{10}	(0.9,0.0)	(0.7,0.1)	(0.8,0.1)	(0.9,0.0)	(0.1,0.8)	(0.8,0.1)	A

According to the concept of granular computing, the three dominance relations can be considered as three granulations [34] with different perspectives. For each of the three dominance relation sets, we have different dominance classes containing numerous attribute subsets. Below, we present another example to illustrate multiple-attribute problems with different focuses.

Example 3.3. As shown in Table 6, we consider an IFODIS containing medical evaluation information. $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ represents ten hospitals from different regions. $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ is composed of six evaluation indexes, where $a_i (i = 1, 2, 3, 4, 5, 6)$ represents equipment, service level, technical level, management level, treatment direction, and hospital scale, respectively. The value of the decision attribute $C \prec B \prec A$ represents the final level in this evaluation.

To obtain three attribute subsets with three different dominance relations, we must determine the dominance relations with respect to each attribute a_i.

The computation of different dominance classes $\widetilde{[x_j]}^{G\geq}_{A_i} (i = 1, 2, 3)$ and the decision dominance class for each object is similar to Example 3.2. Therefore, we have

$$\begin{aligned}
&\widetilde{RC}^{S\geq}_{d|a_1}(U) > \widetilde{RC}^{H\geq}_{d|a_1}(U) > \widetilde{RC}^{T\geq}_{d|a_1}(U);\\
&\widetilde{RC}^{T\geq}_{d|a_2}(U) > \widetilde{RC}^{H\geq}_{d|a_2}(U) > \widetilde{RC}^{S\geq}_{d|a_2}(U);\\
&\widetilde{RC}^{H\geq}_{d|a_3}(U) > \widetilde{RC}^{S\geq}_{d|a_3}(U) > \widetilde{RC}^{T\geq}_{d|a_3}(U);\\
&\widetilde{RC}^{T\geq}_{d|a_4}(U) > \widetilde{RC}^{H\geq}_{d|a_4}(U) > \widetilde{RC}^{S\geq}_{d|a_4}(U);\\
&\widetilde{RC}^{S\geq}_{d|a_5}(U) > \widetilde{RC}^{T\geq}_{d|a_5}(U) > \widetilde{RC}^{H\geq}_{d|a_5}(U);\\
&\widetilde{RC}^{S\geq}_{d|a_6}(U) > \widetilde{RC}^{T\geq}_{d|a_6}(U) > \widetilde{RC}^{H\geq}_{d|a_6}(U).
\end{aligned}$$

From the adjustable-perspective dominance relation in Definition 3.3, we can get

$$\begin{aligned}
&\widetilde{R}^{G_S\geq}_{a_1} = \widetilde{R}^{T\geq}_{a_1}, \widetilde{R}^{G_S\geq}_{a_3} = \widetilde{R}^{T\geq}_{a_3};\\
&\widetilde{R}^{G_S\geq}_{a_2} = \widetilde{R}^{S\geq}_{a_2}, \widetilde{R}^{G_S\geq}_{a_4} = \widetilde{R}^{S\geq}_{a_4};\\
&\widetilde{R}^{G_S\geq}_{a_5} = \widetilde{R}^{H\geq}_{a_5}, \widetilde{R}^{G_S\geq}_{a_6} = \widetilde{R}^{H\geq}_{a_6}.
\end{aligned}$$

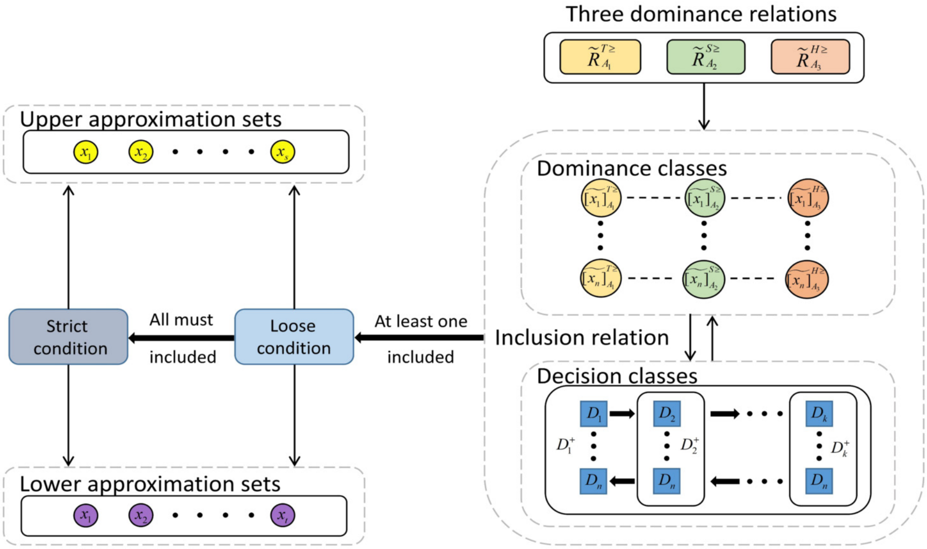


Fig. 5. Research process of DPRS.

Therefore, we can obtain the attribute subsets $A_1 = \{a_1, a_3\}$, $A_2 = \{a_2, a_4\}$, $A_3 = \{a_5, a_6\}$ corresponding to the strong dominance relation, weak dominance relation, and hesitant dominance relation, respectively.

For A_1, A_2, and A_3, the dominance relations of A_1, A_2, A_3 can be demonstrated and verified based on the adjustable-perspective dominance relation. According to Definition 3.3, we are able to acquire

$$\widetilde{R}^{G_S\geq}_{A_1} = \widetilde{R}^{T\geq}_{A_1}, \widetilde{R}^{G_S\geq}_{A_2} = \widetilde{R}^{S\geq}_{A_2}, \widetilde{R}^{G_S\geq}_{A_3} = \widetilde{R}^{H\geq}_{A_3}.$$

The relations of A_1, A_2, and A_3 are determined to be the strong dominance relation, weak dominance relation, and hesitant dominance relation, respectively.

Based on this description, the information table can be divided into granulations with three dominance relations. As shown in Fig. 5, the corresponding dominance classes can be induced from different dominance relations by analyzing the containment relationships between the dominance classes and target decision classes. Specifically, two dissimilar scenarios can be considered: at least one dominance class with the correct perspective must be included in the target concept or all dominance classes with every perspective must be included in the target concept. We first consider the DPRS model under loose conditions.

Definition 3.5. Let $\widetilde{I}^{\geq} = (U, AT \cup \{d\}, V, f)$ be an IFODIS $A_1 \cup A_2 \cup A_3 = AT$, where A_1, A_2, and A_3 represent the attribute sets of the strong, weak, and hesitant dominance relations induced by the adjustable-perspective dominance relation $\widetilde{R}^{G_S\geq}_{A}$, respectively. $\forall X \subseteq U$. The lower and upper approximations of X with respect to $\widetilde{R}^{T\geq}_{A_1}$, $\widetilde{R}^{S\geq}_{A_2}$, and $\widetilde{R}^{H\geq}_{A_3}$ are defined as

$$\underline{\underline{\widetilde{LM}^{G_S\geq}_{A_1+A_2+A_3}}}(X) = \left\{x \in U \,|\, \left([\widetilde{x}]^{T\geq}_{A_1} \subseteq X\right) \vee \left([\widetilde{x}]^{S\geq}_{A_2} \subseteq X\right) \vee \left([\widetilde{x}]^{H\geq}_{A_3} \subseteq X\right)\right\},$$
$$\overline{\widetilde{LM}^{G_S\geq}_{A_1+A_2+A_3}}(X) = \left\{x \in U \,|\, \left([\widetilde{x}]^{T\geq}_{A_1} \cap X \neq \emptyset\right) \wedge \left([\widetilde{x}]^{S\geq}_{A_2} \cap X \neq \emptyset\right) \wedge \left([\widetilde{x}]^{H\geq}_{A_3} \cap X \neq \emptyset\right)\right\},$$

where "$\vee$" and "$\wedge$" signify "or" and "and," respectively. $[\widetilde{x}]^{T\geq}_{A_1} = [\widetilde{x}]^{G_S\geq}_{A_1} = \{y \,|\, (x, y) \in \widetilde{R}^{G_S\geq}_{A_1}\}$ represents the strong dominance class, $[\widetilde{x}]^{S\geq}_{A_2} = [\widetilde{x}]^{G_S\geq}_{A_2} = \{y \,|\, (x, y) \in \widetilde{R}^{G_S\geq}_{A_2}\}$ denotes the weak dominance class, and $[\widetilde{x}]^{H\geq}_{A_3} = [\widetilde{x}]^{G_S\geq}_{A_3} = \{y \,|\, (x, y) \in \widetilde{R}^{G_S\geq}_{A_3}\}$ denotes the hesitant dominance class. $\underline{\widetilde{LM}^{G_S\geq}_{A_1+A_2+A_3}}(X)$ indicates that there is at least one dominance class with the correct perspective included in the target concept. Otherwise, $\overline{\widetilde{LM}^{G_S\geq}_{A_1+A_2+A_3}}(X)$ requires that the intersections of the dominance classes with all perspectives included in the target concept are not empty. If $\underline{\widetilde{LM}^{G_S\geq}_{A_1+A_2+A_3}}(X) \neq \overline{\widetilde{LM}^{G_S\geq}_{A_1+A_2+A_3}}(X)$, then X is called an LDPRS with respect to A_1, A_2, A_3.

Proposition 3.3. *Let $\widetilde{I}^{\geq} = (U, AT \cup \{d\}, V, f)$ be an IFODIS $A_1 \cup A_2 \cup A_3 = AT$. For $X \in P(U)$, we have*

(1) $\underline{\widetilde{LM}^{G_S\geq}_{A_1+A_2+A_3}}(X) \subseteq X \subseteq \overline{\widetilde{LM}^{G_S\geq}_{A_1+A_2+A_3}}(X)$,

(2) $\underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(\sim X) = \sim \overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X)$, $\overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(\sim X) = \sim \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X)$,

(3) $\underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X) = \underline{\widetilde{R}}^{T\geq}_{A_1}(X) \cup \underline{\widetilde{R}}^{S\geq}_{A_2}(X) \cup \underline{\widetilde{R}}^{H\geq}_{A_3}(X)$, $\overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X) = \overline{\widetilde{R}}^{T\geq}_{A_1}(X) \cap \overline{\widetilde{R}}^{S\geq}_{A_2}(X) \cap \overline{\widetilde{R}}^{H\geq}_{A_3}(X)$.

Proof. The theorems above can be directly derived from Definition 3.5.

Proposition 3.4. *Let $\widetilde{I}^{\geq} = (U, AT \cup \{d\}, V, f)$ be an IFODIS, $A_1 \cup A_2 \cup A_3 = AT$. For $X, Y \in P(U)$, we have*

(1) $\underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X \cap Y) \subseteq \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X) \cap \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(Y)$, $\overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X \cup Y) \supseteq \overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X) \cup \overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(Y)$,

(2) $\underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X \cup Y) \supseteq \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X) \cup \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(Y)$, $\overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X \cap Y) \subseteq \overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X) \cap \overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(Y)$,

(3) $\underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X \cap Y) = (\underline{\widetilde{R}}^{T\geq}_{A_1}(X) \cap \underline{\widetilde{R}}^{T\geq}_{A_1}(Y)) \cup (\underline{\widetilde{R}}^{S\geq}_{A_2}(X) \cap \underline{\widetilde{R}}^{S\geq}_{A_2}(Y)) \cup (\underline{\widetilde{R}}^{H\geq}_{A_3}(X) \cap \underline{\widetilde{R}}^{H\geq}_{A_3}(Y))$,

(4) $\overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X \cup Y) = (\underline{\widetilde{R}}^{T\geq}_{A_1}(X) \cup \underline{\widetilde{R}}^{T\geq}_{A_1}(Y)) \cap (\underline{\widetilde{R}}^{S\geq}_{A_2}(X) \cup \underline{\widetilde{R}}^{S\geq}_{A_2}(Y)) \cap (\underline{\widetilde{R}}^{H\geq}_{A_3}(X) \cup \underline{\widetilde{R}}^{H\geq}_{A_3}(Y))$.

Proof. (1) For $\forall x \in \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X \cap Y)$, we have $[\widetilde{x}]^{T\geq}_{A_1} \subseteq (X \cap Y)$ or $[\widetilde{x}]^{S\geq}_{A_2} \subseteq (X \cap Y)$ or $[\widetilde{x}]^{H\geq}_{A_3} \subseteq (X \cap Y)$ according to Definition 3.5. Then, we know that $[\widetilde{x}]^{T\geq}_{A_1} \subseteq X$ or $[\widetilde{x}]^{S\geq}_{A_2} \subseteq X$ or $[\widetilde{x}]^{H\geq}_{A_3} \subseteq X$ holds. $[\widetilde{x}]^{T\geq}_{A_1} \subseteq Y$ or $[\widetilde{x}]^{S\geq}_{A_2} \subseteq Y$ or $[\widetilde{x}]^{H\geq}_{A_3} \subseteq Y$ also holds. Therefore, $x \in \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X)$ and $x \in \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(Y)$ hold, meaning $x \in \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X) \cap \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(Y)$. As a result, $\underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X \cap Y) \subseteq \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X) \cap \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(Y)$. Similarly, we can obtain $\overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X \cup Y) \supseteq \overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X) \cup \overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(Y)$.

(2) Because $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, we have that $\underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X) \subseteq \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X \cap Y)$, $\underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(Y) \subseteq \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X \cap Y)$. Therefore, $\underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X) \cup \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(Y) \subseteq \underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X \cap Y)$. Similarly, $\overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X \cap Y) \subseteq \overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(X) \cap \overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(Y)$ can be proved.

(3) $\sim$ (4) This proof can be obtained from Proposition 3.3.

In the LDPRS, the three dominance classes only require that at least one type of dominance class is included in the target concept to reflect relatively loose conditions. In contrast, when the conditions require that all three types of dominance classes must be included in the target concept, then we have the TDARS, which we will investigate next.

Definition 3.6. Let $\widetilde{I}^{\geq} = (U, AT \cup \{d\}, V, f)$ be an IFODIS $A_1 \cup A_2 \cup A_3 = AT$, where A_1, A_2, and A_3 represent the attribute sets of the strong, weak, and hesitant dominance relations induced by the adjustable-perspective dominance relation $\widetilde{R}^{G_S\geq}_{A}$, respectively. $\forall X \subseteq U$. The lower and upper approximations of X with respect to $\widetilde{R}^{T\geq}_{A_1}$, namely $\widetilde{R}^{S\geq}_{A_2}$ and $\widetilde{R}^{H\geq}_{A_3}$, are defined as

$$\underline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(X) = \left\{x \in U | \left([\widetilde{x}]^{T\geq}_{A_1} \subseteq X\right) \wedge \left([\widetilde{x}]^{S\geq}_{A_2} \subseteq X\right) \wedge \left([\widetilde{x}]^{H\geq}_{A_3} \subseteq X\right)\right\},$$
$$\overline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(X) = \left\{x \in U | \left([\widetilde{x}]^{T\geq}_{A_1} \cap X \neq \emptyset\right) \vee \left([\widetilde{x}]^{S\geq}_{A_2} \cap X \neq \emptyset\right) \vee \left([\widetilde{x}]^{H\geq}_{A_3} \cap X \neq \emptyset\right)\right\},$$

where "$\vee$" denotes "or" and "$\wedge$" denotes "and." $[\widetilde{x}]^{T\geq}_{A_1} = [\widetilde{x}]^{G_S\geq}_{A_1} = \{y | (x, y) \in \widetilde{R}^{G_S\geq}_{A_1}\}$, $[\widetilde{x}]^{S\geq}_{A_2} = [\widetilde{x}]^{G_S\geq}_{A_2} = \{y | (x, y) \in \widetilde{R}^{G_S\geq}_{A_1}\}$, and $[\widetilde{x}]^{H\geq}_{A_3} = \{y | (x, y) \in \widetilde{R}^{G_S\geq}_{A_1}\}$ represent with the strong, weak, and hesitant dominance classes, respectively. $\underline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(X)$ indicates that all dominance classes with every specific perspective are included in the target concept. Otherwise, $\overline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(X)$ indicates that the intersection of at least one type of dominance class and the target concept is not empty. If $\underline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(X) \neq \overline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(X)$, then X is called an SDPRS with respect to A_1, A_2, A_3.

Proposition 3.5. *Let $\widetilde{I}^{\geq} = (U, AT \cup \{d\}, V, f)$ be an IFODIS $A_1 \cup A_2 \cup A_3 = AT$. For $X \in P(U)$, we have*

(1) $\underline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(X) \subseteq X \subseteq \overline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(X)$,

(2) $\underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(\sim X) = \sim \overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X)$, $\overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(\sim X) = \sim \underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X)$,

(3) $\underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X) = \underline{\widetilde{R}^{T\geq}_{A_1}}(X) \cap \underline{\widetilde{R}^{S\geq}_{A_2}}(X) \cap \underline{\widetilde{R}^{H\geq}_{A_3}}(X)$, $\overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X) = \overline{\widetilde{R}^{T\geq}_{A_1}}(X) \cup \overline{\widetilde{R}^{S\geq}_{A_2}}(X) \cup \overline{\widetilde{R}^{H\geq}_{A_3}}(X)$.

Proof. The corresponding proof can be obtained directly from Definition 3.6.

Proposition 3.6. *Let $\widetilde{I}^{\geq} = (U, AT \cup \{d\}, V, f)$ be an IFODIS $A_1 \cup A_2 \cup A_3 = AT$. For $X, Y \in P(U)$, we have*

(1) $\underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X \cap Y) = \underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X) \cap \underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(Y)$, $\overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X \cap Y) \subseteq \overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X) \cap \overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(Y)$,

(2) $\overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X \cup Y) = \overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X) \cap \overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(Y)$, $\underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X \cup Y) \supseteq \underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X) \cup \underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(Y)$,

(3) $\underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X \cap Y) = (\underline{\widetilde{R}^{T\geq}_{A_1}}(X) \cap \underline{\widetilde{R}^{T\geq}_{A_1}}(Y)) \cap (\underline{\widetilde{R}^{S\geq}_{A_2}}(X) \cap \underline{\widetilde{R}^{S\geq}_{A_2}}(Y)) \cap (\underline{\widetilde{R}^{H\geq}_{A_3}}(X) \cap \underline{\widetilde{R}^{H\geq}_{A_3}}(Y))$,

(4) $\overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X \cup Y) = (\underline{\widetilde{R}^{T\geq}_{A_1}}(X) \cup \underline{\widetilde{R}^{T\geq}_{A_1}}(Y)) \cup (\underline{\widetilde{R}^{S\geq}_{A_2}}(X) \cup \underline{\widetilde{R}^{S\geq}_{A_2}}(Y)) \cup (\underline{\widetilde{R}^{H\geq}_{A_3}}(X) \cup \underline{\widetilde{R}^{H\geq}_{A_3}}(Y))$.

Proof. (1) For $x \in \underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X \cap Y)$, we have that

$$\begin{aligned} x \in \underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X \cap Y) &\Leftrightarrow \widetilde{[x]}^{T\geq}_{A_1+A_2+A_3} \subseteq (X \cap Y) \wedge \widetilde{[x]}^{S\geq}_{A_1+A_2+A_3} \subseteq (X \cap Y) \wedge \widetilde{[x]}^{H\geq}_{A_1+A_2+A_3} \subseteq (X \cap Y) \\ &\Leftrightarrow \widetilde{[x]}^{T\geq}_{A_1+A_2+A_3} \subseteq X \wedge \widetilde{[x]}^{S\geq}_{A_1+A_2+A_3} \subseteq X \wedge \widetilde{[x]}^{H\geq}_{A_1+A_2+A_3} \subseteq X, \\ &\quad and\ \widetilde{[x]}^{T\geq}_{A_1+A_2+A_3} \subseteq Y \wedge \widetilde{[x]}^{S\geq}_{A_1+A_2+A_3} \subseteq Y \wedge \widetilde{[x]}^{H\geq}_{A_1+A_2+A_3} \subseteq Y, \\ &\Leftrightarrow x \in \underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X) \cap x \in \underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(Y). \end{aligned}$$

Therefore, $\underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X \cap Y) = \underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X) \cap \underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(Y)$ holds.

Because $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, we have that $\overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X \cap Y) \subseteq \overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X)$, $\overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X \cap Y) \subseteq \overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(Y)$. Therefore, $\overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X \cap Y) \subseteq \overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X) \cap \overline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(Y)$ holds.

(2) The proof is similar to (1).

(3) $\sim$ (4) This proof can be obtained from Proposition 3.5.

In the classical dominance rough set theory, decision makers apply the lower and upper approximations of concept X in dominance relation $\widetilde{R}^{G_S\geq}_{A}$ to extract dominance rules. Deterministic dominance rules can be acquired from the lower approximation and possibility dominance rules can be obtained from the boundary domain. In combination with the concept of the different-perspective dominance relation, two types of decision rules with logical meanings of "or" and "and" can be extracted from the SPRS constructed above. $T(x,a), S(x,a), H(x,a)$ are abbreviated as $T_a(x), S_a(x), H_a(x)$, respectively. $a_i \subseteq A_1 (i = t, \cdots, s)$, $a_j \subseteq A_2 (j = m, \cdots, n)$, $a_v \subseteq A_3 (v = l, \cdots, k)$. $A_1, A_2, A_3 \subseteq AT$.
The specific forms are defined as follows:

- "At least":

(1) $\big((T_{a_1}(x) \geq \nu_{a_1}) \wedge (T_{a_3}(x) \geq \nu_{a_3}) \wedge \cdots \wedge (T_{a_s}(x) \geq \nu_{a_s})\big) \vee \big((S_{a_2}(x) \geq \nu_{a_2}) \wedge (S_{a_4}(x) \geq \nu_{a_4}) \wedge \cdots \wedge (S_{a_n}(x) \geq \nu_{a_n})\big) \vee \big((H_{a_5}(x) \geq \nu_{a_5}) \wedge (H_{a_6}(x) \geq \nu_{a_6}) \wedge \cdots \wedge (H_{a_k}(x) \geq \nu_{a_k})\big) \rightarrow (d \geq C)$,

(2) $\big((T_{a_1}(x) \leq \nu_{a_1}) \wedge (T_{a_3}(x) \leq \nu_{a_3}) \wedge \cdots \wedge (T_{a_s}(x) \leq \nu_{a_s})\big) \wedge \big((S_{a_2}(x) \leq \nu_{a_2}) \wedge (S_{a_4}(x) \leq \nu_{a_4}) \wedge \cdots \wedge (S_{a_n}(x) \leq \nu_{a_n})\big) \wedge \big((H_{a_5}(x) \leq \nu_{a_5}) \wedge (H_{a_6}(x) \leq \nu_{a_6}) \wedge \cdots \wedge (H_{a_k}(x) \leq \nu_{a_k})\big) \rightarrow (d \leq A)$.

- "At most":

(3) $\big((T_{a_1}(x) \geq \nu_{a_1}) \wedge (T_{a_3}(x) \geq \nu_{a_3}) \wedge \cdots \wedge (T_{a_s}(x) \geq \nu_{a_s})\big) \wedge \big((S_{a_2}(x) \geq \nu_{a_2}) \wedge (S_{a_4}(x) \geq \nu_{a_4}) \wedge \cdots \wedge (S_{a_n}(x) \geq \nu_{a_n})\big) \wedge \big((H_{a_5}(x) \geq \nu_{a_5}) \wedge (H_{a_6}(x) \geq \nu_{a_6}) \wedge \cdots \wedge (H_{a_k}(x) \geq \nu_{a_k})\big) \rightarrow (d \geq C)$,

(4) $\big((T_{a_1}(x) \leq \nu_{a_1}) \wedge (T_{a_3}(x) \leq \nu_{a_3}) \wedge \cdots \wedge (T_{a_s}(x) \leq \nu_{a_s})\big) \vee \big((S_{a_2}(x) \leq \nu_{a_2}) \wedge (S_{a_4}(x) \leq \nu_{a_4}) \wedge \cdots \wedge (S_{a_n}(x) \leq \nu_{a_n})\big) \vee \big((H_{a_5}(x) \leq \nu_{a_5}) \wedge (H_{a_6}(x) \leq \nu_{a_6}) \wedge \cdots \wedge (H_{a_k}(x) \leq \nu_{a_k})\big) \rightarrow (d \leq A)$.

Example 3.4. (Continued from Example 3.3) Because we have obtained the dominance relations of attribute sets $A_1 = \{a_1, a_3\}$, $A_2 = \{a_2, a_4\}$ and $A_3 = \{a_5, a_6\}$, we only present the corresponding dominance class for each attribute set in Table 6.

The strong dominance classes with respect to A_1 are also calculated.

$$\widetilde{[x_1]}_{A_1}^{T\geq} = \{x_1\}; \quad \widetilde{[x_6]}_{A_1}^{T\geq} = \{x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}\};$$
$$\widetilde{[x_2]}_{A_1}^{T\geq} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}\}; \quad \widetilde{[x_7]}_{A_1}^{T\geq} = \{x_1, x_5, x_7, x_{10}\};$$
$$\widetilde{[x_3]}_{A_1}^{T\geq} = \{x_1, x_3, x_4, x_5, x_7, x_8, x_{10}\}; \quad \widetilde{[x_8]}_{A_1}^{T\geq} = \{x_1, x_3, x_4, x_5, x_7, x_8, x_{10}\};$$
$$\widetilde{[x_4]}_{A_1}^{T\geq} = \{x_1, x_3, x_4, x_5, x_7, x_8, x_{10}\}; \quad \widetilde{[x_9]}_{A_1}^{T\geq} = \{x_1, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\};$$
$$\widetilde{[x_5]}_{A_1}^{T\geq} = \{x_1, x_5, x_{10}\}; \quad \widetilde{[x_{10}]}_{A_1}^{T\geq} = \{x_1, x_{10}\}.$$

The weak dominance classes with respect to A_2 are calculated.

$$\widetilde{[x_1]}_{A_2}^{S\geq} = \{x_1\}; \quad \widetilde{[x_6]}_{A_2}^{S\geq} = \{x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}\};$$
$$\widetilde{[x_2]}_{A_2}^{S\geq} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}\}; \quad \widetilde{[x_7]}_{A_2}^{S\geq} = \{x_1, x_5, x_7, x_{10}\};$$
$$\widetilde{[x_3]}_{A_2}^{S\geq} = \{x_1, x_3, x_5, x_7, x_8, x_{10}\}; \quad \widetilde{[x_8]}_{A_2}^{S\geq} = \{x_1, x_5, x_7, x_8, x_{10}\};$$
$$\widetilde{[x_4]}_{A_2}^{S\geq} = \{x_1, x_3, x_4, x_5, x_7, x_8, x_{10}\}; \quad \widetilde{[x_9]}_{A_2}^{S\geq} = \{x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\};$$
$$\widetilde{[x_5]}_{A_2}^{S\geq} = \{x_1, x_5, x_{10}\}; \quad \widetilde{[x_{10}]}_{A_2}^{S\geq} = \{x_{10}\}.$$

The hesitant dominance classes with respect to A_3 are calculated.

$$\widetilde{[x_1]}_{A_3}^{H\geq} = \{x_1, x_5, x_{10}\}; \quad \widetilde{[x_6]}_{A_3}^{H\geq} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}\};$$
$$\widetilde{[x_2]}_{A_3}^{H\geq} = \{x_1, x_2, x_3, x_4, x_5, x_7, x_8, x_{10}\}; \quad \widetilde{[x_7]}_{A_3}^{H\geq} = \{x_1, x_3, x_4, x_5, x_7, x_8, x_{10}\};$$
$$\widetilde{[x_3]}_{A_3}^{H\geq} = \{x_1, x_3, x_4, x_5, x_8, x_{10}\}; \quad \widetilde{[x_8]}_{A_3}^{H\geq} = \{x_1, x_3, x_4, x_5, x_8, x_{10}\};$$
$$\widetilde{[x_4]}_{A_3}^{H\geq} = \{x_1, x_3, x_4, x_5, x_8, x_{10}\}; \quad \widetilde{[x_9]}_{A_3}^{H\geq} = \{x_1, x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\};$$
$$\widetilde{[x_5]}_{A_3}^{H\geq} = \{x_1, x_5, x_{10}\}; \quad \widetilde{[x_{10}]}_{A_3}^{H\geq} = \{x_1, x_5, x_{10}\}.$$

Suppose that $X = \{x_1, x_3, x_5, x_7, x_8, x_{10}\}$, we can figure the lower and upper approximations of X in IFODIS sequentially. According to the above Definition 3.5, we compute readily the lower and upper approximations:

$$\underline{\underline{\widetilde{LM}_{A_1+A_2+A_3}^{G_S\geq}}}(X) = \{x_1, x_5, x_7, x_8, x_{10}\};$$
$$\overline{\overline{\widetilde{LM}_{A_1+A_2+A_3}^{G_S\geq}}}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}.$$

Similarly, we are able to compute the approximations in the strict situation:

$$\underline{\underline{\widetilde{TM}_{A_1+A_2+A_3}^{G_S\geq}}}(X) = \{x_1, x_5, x_{10}\};$$
$$\overline{\overline{\widetilde{TM}_{A_1+A_2+A_3}^{G_S\geq}}}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}.$$

From approximation results, two types of decision rules can be generated, here we only introduce the deterministic dominance rules from the lower approximations

$\underline{\widetilde{LM}_{A_1+A_2+A_3}^{G_S\geq}}(X)$:

$$\big((T_{a_1}(x) \geq 0.9) \wedge (T_{a_3}(x) \geq 0.9)\big) \vee \big((S_{a_2}(x) \geq 1.0) \wedge (S_{a_4}(x) \geq 0.9)\big) \vee \big((H_{a_5}(x) \geq -0.1) \wedge (H_{a_6}(x) \geq -0.1)\big) \to (d \geq A);$$

$$\big((T_{a_1}(x) \geq 0.7) \wedge (T_{a_3}(x) \geq 0.7)\big) \vee \big((S_{a_2}(x) \geq 0.9) \wedge (S_{a_4}(x) \geq 0.9)\big) \vee \big((H_{a_5}(x) \geq -0.1) \wedge (H_{a_6}(x) \geq -0.1)\big) \to (d \geq A);$$

$$\big((T_{a_1}(x) \geq 0.6) \wedge (T_{a_3}(x) \geq 0.6)\big) \vee \big((S_{a_2}(x) \geq 0.7) \wedge (S_{a_4}(x) \geq 0.8)\big) \vee \big((H_{a_5}(x) \geq -0.3) \wedge (H_{a_6}(x) \geq -0.2)\big) \to (d \geq B);$$

$$\big((T_{a_1}(x) \geq 0.5) \wedge (T_{a_3}(x) \geq 0.6)\big) \vee \big((S_{a_2}(x) \geq 0.7) \wedge (S_{a_4}(x) \geq 0.7)\big) \vee \big((H_{a_5}(x) \geq -0.2) \wedge (H_{a_6}(x) \geq -0.2)\big) \to (d \geq B);$$

$$\big((T_{a_1}(x) \geq 0.9) \wedge (T_{a_3}(x) \geq 0.8)\big) \vee \big((S_{a_2}(x) \geq 0.9) \wedge (S_{a_4}(x) \geq 1.0)\big) \vee \big((H_{a_5}(x) \geq -0.1) \wedge (H_{a_6}(x) \geq -0.1)\big) \to (d \geq A).$$

$\underline{\widetilde{TM}_{A_1+A_2+A_3}^{G_S\geq}}(X)$:

$$\big((T_{a_1}(x) \geq 0.9) \wedge (T_{a_3}(x) \geq 0.9)\big) \wedge \big((S_{a_2}(x) \geq 1.0) \wedge (S_{a_4}(x) \geq 0.9)\big) \wedge \big((H_{a_5}(x) \geq -0.1) \wedge (H_{a_6}(x) \geq -0.1)\big) \to (d \geq A);$$

Table 7
The time complexity of SPRS and DPRS.

Steps	SPRS	Steps	LDPRS	TDPRS
	Algorithm 2		Algorithms 2 and 3	Algorithms 2 and 3
1-16	$O(\lvert U\rvert)$	1-17	$O(\lvert AT\rvert\lvert U\rvert)$	$O(\lvert AT\rvert\lvert U\rvert)$
	Algorithm 1		Algorithm 4	Algorithm 4
1-8	$O(\lvert AT\rvert\lvert U\rvert^2)$	1-5	$O(\lvert AT\rvert\lvert U\rvert^2)$	$O(\lvert AT\rvert\lvert U\rvert^2)$
9-17	$O(\lvert AT\rvert\lvert D\rvert\lvert U\rvert)$	6-14	$O(\lvert D\rvert\lvert U\rvert)$	$O(\lvert D\rvert\lvert U\rvert)$
18-21	$O(\lvert AT\rvert\lvert D\rvert)$	15-17	$O(\lvert D\rvert)$	$O(\lvert D\rvert)$
Total	$O(\lvert AT\rvert\lvert U\rvert^2)+\lvert AT\rvert\lvert D\rvert\lvert U\rvert+\lvert AT\rvert\lvert D\rvert)$	Total	$O(\lvert AT\rvert\lvert U\rvert^2+\lvert D\rvert\lvert U\rvert+\lvert D\rvert)$	$O(\lvert AT\rvert\lvert U\rvert^2+\lvert D\rvert\lvert U\rvert+\lvert D\rvert)$

$$\big((T_{a_1}(x)\geq 0.7)\wedge(T_{a_3}(x)\geq 0.7)\big)\wedge\big((S_{a_2}(x)\geq 0.9)\wedge(S_{a_4}(x)\geq 0.9)\big)\wedge\big((H_{a_5}(x)\geq -0.1)\wedge(H_{a_6}(x)\geq -0.1)\big)\to(d\geq A);$$

$$\big((T_{a_1}(x)\geq 0.9)\wedge(T_{a_3}(x)\geq 0.8)\big)\wedge\big((S_{a_2}(x)\geq 0.9)\wedge(S_{a_4}(x)\geq 1.0)\big)\wedge\big((H_{a_5}(x)\geq -0.1)\wedge(H_{a_6}(x)\geq -0.1)\big)\to(d\geq A).$$

Based on the deterministic dominance rules, one can see that $\langle T_{a_1}(x), T_{a_3}(x)\rangle$, $\langle S_{a_2}(x), S_{a_4}(x)\rangle$, and $\langle H_{a_5}(x), H_{a_6}(x)\rangle$ represent three groups. Following the previous calculation, we can obtain that attribute sets A_1, A_2, and A_3 focus on membership, non-membership, and hesitation, respectively. Under the deterministic rules of $\underline{\widetilde{LM}^{G_S\geq}_{A_1+A_2+A_3}}(X)$ and $\underline{\widetilde{TM}^{G_S\geq}_{A_1+A_2+A_3}}(X)$, $\forall x\in U$ must satisfy

$$\big((\mu_{a_1}(x)\geq 0.6)\wedge(\mu_{a_3}(x)\geq 0.6)\big)\wedge\big((\nu_{a_2}(x)\leq 0.3)\wedge(\nu_{a_4}(x)\leq 0.2)\big)\wedge\big((\omega_{a_5}(x)\leq 0.3)\wedge(\omega_{a_6}(x)\leq 0.2)\big)\to(d\geq B);$$

$$\big((\mu_{a_1}(x)\geq 0.5)\wedge(\mu_{a_3}(x)\geq 0.6)\big)\wedge\big((\nu_{a_2}(x)\leq 0.3)\wedge(\nu_{a_4}(x)\leq 0.3)\big)\wedge\big((\omega_{a_5}(x)\leq 0.2)\wedge(\omega_{a_6}(x)\leq 0.2)\big)\to(d\geq B).$$

Hospital x will receive a grade of at least B in this medical evaluation. When $\forall x\in U$ satisfies

$$\big((\mu_{a_1}(x)\geq 0.9)\wedge(\mu_{a_3}(x)\geq 0.9)\big)\wedge\big((\nu_{a_2}(x)\leq 0.0)\wedge(\nu_{a_4}(x)\leq 0.1)\big)\wedge\big((\omega_{a_5}(x)\leq 0.1)\wedge(\omega_{a_6}(x)\leq 0.1)\big)\to(d\geq A),$$

$$\big((\mu_{a_1}(x)\geq 0.7)\wedge(\mu_{a_3}(x)\geq 0.7)\big)\wedge\big((\nu_{a_2}(x)\leq 0.1)\wedge(\nu_{a_4}(x)\leq 0.1)\big)\wedge\big((\omega_{a_5}(x)\leq 0.1)\wedge(\omega_{a_6}(x)\leq 0.1)\big)\to(d\geq A),$$

$$\big((\mu_{a_1}(x)\geq 0.9)\wedge(\mu_{a_3}(x)\geq 0.8)\big)\wedge\big((\nu_{a_2}(x)\leq 0.1)\wedge(\nu_{a_4}(x)\leq 0.0)\big)\wedge\big((\omega_{a_5}(x)\leq 0.1)\wedge(\omega_{a_6}(x)\leq 0.1)\big)\to(d\geq A).$$

Then hospital x must receive a grade of A in this medical evaluation.

3.3. Algorithms for computing the roughness and dependence degree in IFODIS

In this subsection, we present four algorithms for calculating the upper and lower approximations based on the adjustable-perspective dominance relation and other five dominance relations, and compare them based on the indicators of roughness and dependence degree.

In Algorithm 1, an IFODIS is inputted in the first step and then the second step computes all decision classes $U/d = \{D_1, D_2, ..., D_m\}$. Steps 3 to 6 calculate all types of dominance classes of every x_i in the IFODIS. Steps 7 and 8 initialize the lower and upper approximations of each dominance class. Steps 9 to 20 obtain the roughness and dependence degrees of all dominance classes with respect to each decision class. Finally, steps 21 to 23 return the roughness and the dependence degrees in the IFODIS.

In the process of computing the dominance class, the adjustable-perspective dominance relation from Algorithm 2 is required. Algorithm 2 is used to determine the characteristic of an attribute while implementing the process of the adjustable-perspective dominance relation. First, Steps 1 to 4 acquire three dominance classes and decision classes according to Definition 3.1. Step 5 calculates the dominance conditional entropy under different dominance relations. Steps 6 to 14 compute the adjustable-perspective dominance relation. Finally, Step 15 returns the adjustable-perspective dominance relation with respect to A_t.

By using Algorithm 2, we are able to determine the characteristic of an attribute. Therefore, based on Algorithm 2, all attributes in an IFODIS can be divided into three types of attribute sets. Algorithm 3 describes the generation of three attribute subsets. In Algorithm 3, Step 2 initializes the attribute subsets. Steps 3 to 14 separate three different attribute subsets from AT. Step 15 acquires strong, weak, and hesitant dominance relations. Finally, Step 17 returns three types of attribute subsets and the corresponding strong, weak, and hesitant dominance relations.

By combining these algorithms, the attributes of an IFODIS are divided into three attribute subsets with different emphases. Algorithm 4 implements the computation of the roughness and dependence degree based on the three attribute subsets. First, Steps 1 to 4 initialize the lower and upper approximations of LDPRS and TDPRS. In Steps 5 to 10, three types of support characteristic functions are calculated based on Definitions 3.5 and 3.6. Then, Steps 11 to 25 compute the lower and upper approximations of three rough set models with respect to different decision classes. Steps 26 to 29 compute the roughness and dependence degree of the three rough set models. Finally, Step 30 returns the roughness and dependence degree of the LDPRS and TDPRS in an IFODIS.

Algorithm 1: The algorithm for computing the roughness and the dependence degree of SPRS in IFODIS.

Input : An IFODIS $\widetilde{I}^{\geq} = (U, AT \cup d, V, f)$, attribute subsets $A_t \subseteq AT(t = 1, 2, ..., s)$.
Output : The roughness and the dependence degree of SPRS in IFODIS

begin
  Compute $U/d = \{D_1, D_2, \cdots, D_N\}$
  for $t = 1$ *to* s **do**
    for $i = 1$ *to* $|U|$ **do**
      Compute $\widehat{[x_i]}^{\geq}_{A_t}, \widetilde{[x_i]}^{\geq}_{A_t}, \widetilde{[x_i]}^{T\geq}_{A_t}, \widetilde{[x_i]}^{S\geq}_{A_t}, \widetilde{[x_i]}^{H\geq}_{A_t}, \widetilde{[x_i]}^{G_S\geq}_{A_t}(I \sim VI)$;
    end
  end
  Let $\underline{\underline{\widehat{R}^{\geq}_{A_t}}}(D_j), \underline{\underline{\widetilde{R}^{\geq}_{A_t}}}(D_j), \underline{\underline{\widetilde{R}^{T\geq}_{A_t}}}(D_j), \underline{\underline{\widetilde{R}^{S\geq}_{A_t}}}(D_j), \underline{\underline{\widetilde{R}^{H\geq}_{A_t}}}(D_j), \underline{\underline{\widetilde{R}^{G_S\geq}_{A_t}}}(D_j)$;
  $\overline{\overline{\widehat{R}^{\geq}_{A_t}}}(D_j), \overline{\overline{\widetilde{R}^{\geq}_{A_t}}}(D_j), \overline{\overline{\widetilde{R}^{T\geq}_{A_t}}}(D_j), \overline{\overline{\widetilde{R}^{S\geq}_{A_t}}}(D_j), \overline{\overline{\widetilde{R}^{H\geq}_{A_t}}}(D_j), \overline{\overline{\widetilde{R}^{G_S\geq}_{A_t}}}(D_j)$.
  for $j = 1$ *to* N **do**
    for $i = 1$ *to* $|U|$ **do**
      if $[x_i]^{\geq}_{A_t} \subseteq D_j (I \sim VI)$ **then**
        $\underline{R}^{\geq}_{A_t}(D_j) = \underline{R}^{\geq}_{A_t}(D_j) \cup \{x_i\}(I \sim VI)$;
      end
      if $[x_i]^{\geq}_{A_t} \cap D_j \neq \emptyset$ **then**
        $\overline{R}^{\geq}_{A_t}(D_j) = \overline{R}^{\geq}_{A_t}(D_j) \cup \{x_i\}(I \sim VI)$;
      end
    end
    $\rho^{\geq}_{A_t}(D_j) = 1 - \dfrac{\underline{\underline{R^{\geq}_{A_t}}}(D_j)}{\overline{\overline{R^{\geq}_{A_t}}}(D_j)}(I \sim VI)$;
    $\gamma_{A_t}(D^+) = \dfrac{\sum_{i=1}^{|N|} \underline{R}^{\geq}_{A_t}(D_j^+)}{\sum_{i=1}^{|N|} \left|D_j^+\right|}(I \sim VI)$;
  end
  return: $\rho^{\geq}_{A_t}(D_j)(I \sim VI)$, $\gamma_{A_t}(D^+)(I \sim VI)$.
end

Algorithm 2: The algorithm to determine the characteristic of an attribute in IFODIS.

Input : An IFODIS $\widetilde{I}^{\geq} = (U, AT \cup d, V, f)$, $A_t \subseteq AT(t = 1, 2, ..., s)$. $U/d = \{D_1, D_2, \cdots, D_N\}$
Output : The adjustable-perspective dominance relation with respect to A_t

begin
  for $i = 1$ *to* $|U|$ **do**
    Compute $\widetilde{[x_i]}^{G\geq}_{A_t}(G \in \{T, S, H\})$ and $D_d^+(x_i)$ by Definition 3.1;
  end
  Compute $\widetilde{RC}^{T\geq}_{d|A}(U), \widetilde{RC}^{S\geq}_{d|A}(U), \widetilde{RC}^{H\geq}_{d|A}(U)$ via using Definition 3.3;
  if $\widetilde{RC}^{T\geq}_{d|A}(U) = \min\limits_{G \in \{T,S,H\}} \widetilde{RC}^{G\geq}_{d|A}(U)$ **then**
    $\widetilde{R}^{G_S\geq}_{A_t} = \widetilde{R}^{T\geq}_{A_t}$
  end
  if $\widetilde{RC}^{S\geq}_{d|A}(U) = \min\limits_{G \in \{T,S,H\}} \widetilde{RC}^{G\geq}_{d|A}(U)$ **then**
    $\widetilde{R}^{G_S\geq}_{A_t} = \widetilde{R}^{S\geq}_{A_t}$
  end
  if $\widetilde{RC}^{H\geq}_{d|A}(U) = \min\limits_{G \in \{T,S,H\}} \widetilde{RC}^{G\geq}_{d|A}(U)$ **then**
    $\widetilde{R}^{G_S\geq}_{A_t} = \widetilde{R}^{H\geq}_{A_t}$
  end
  return: $\widetilde{R}^{G_S\geq}_{A_t}$.
end

Based on the time complexities of Algorithms 1 and 4 in Table 7, we know that the time complexities of the LDPRS and TDPRS are similar. In the computation of the DPRS based on Algorithms 2 and 3, because Algorithm 3 obtains the dominance relation attribute subsets, the approximations of the DPRS ignore the traversal of attributes, meaning the total time complexity of the DPRS is less than that of the SPRS.

4. Experimental studies

In this section, a series of experiments are presented to verify the effectiveness and applicability of the proposed models based on the ordinal classification accuracy, roughness, and dependence degree of the algorithms described above. The

Algorithm 3: The algorithm to determine the characteristic of all attributes in IFODIS.

Input : An IFODIS $\widetilde{I}^{\geq}=(U, AT\cup d, V, f)$, $a_t\in AT(t=1,...,s)$;
Output : Attribute sets A_1, A_2, A_3; and dominance relations $\widetilde{R}^{T\geq}_{A_1}$, $\widetilde{R}^{S\geq}_{A_2}$, $\widetilde{R}^{H\geq}_{A_3}$.

begin
  Initialize $A_1\leftarrow\emptyset$, $A_2\leftarrow\emptyset$, $A_3\leftarrow\emptyset$.
  for $t=1$ *to* s **do**
    Return to Algorithm 2 and calculate $\widetilde{R}^{G_S\geq}_{a_t}$;
    if $\widetilde{R}^{T\geq}_{a_t}=\widetilde{R}^{G_S\geq}_{a_t}$ **then**
      $A_1\leftarrow A_1\cup\{a_t\}$;
    end
    if $\widetilde{R}^{S\geq}_{a_t}=\widetilde{R}^{G_S\geq}_{a_t}$ **then**
      $A_2\leftarrow A_2\cup\{a_t\}$;
    end
    if $\widetilde{R}^{H\geq}_{a_t}=\widetilde{R}^{G_S\geq}_{a_t}$ **then**
      $A_3\leftarrow A_3\cup\{a_t\}$;
    end
  end
  $\widetilde{R}^{T\geq}_{A_1}=\widetilde{R}^{G_S\geq}_{A_1}$, $\widetilde{R}^{S\geq}_{A_2}=\widetilde{R}^{G_S\geq}_{A_2}$, $\widetilde{R}^{H\geq}_{A_3}=\widetilde{R}^{G_S\geq}_{A_3}$.
end
return: A_1, A_2, A_3; $\widetilde{R}^{T\geq}_{A_1}$, $\widetilde{R}^{S\geq}_{A_2}$, $\widetilde{R}^{H\geq}_{A_3}$.

Algorithm 4: The algorithm for computing the roughness and the dependence degree of LDPRS, TDPRS in IFODIS.

Input : An IFODIS $\widetilde{I}^{\geq}=(U, AT\cup d, V, f)$; $A_1, A_2, A_3\subseteq AT$, the quotient set $U/d=\{D_1, D_2, \cdots, D_N\}$.
Output : Strong, weak and hesitant dominance relations with respect to A_1, A_2, A_3.

begin
  $\underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)\leftarrow\emptyset$, $\overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)\leftarrow\emptyset$; $\underline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)\leftarrow\emptyset$, $\overline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)\leftarrow\emptyset$;
  for $i=1$ *to* $|U|$ **do**
    Compute $\widetilde{[x_i]}^{T\geq}_{A_1}$, $\widetilde{[x_i]}^{S\geq}_{A_2}$, $\widetilde{[x_i]}^{H\geq}_{A_1}$ according to Algorithm 3;
  end
  for $j=1$ *to* N **do**
    for $i=1$ *to* $|U|$ **do**
      if $\left(\widetilde{[x_i]}^{T\geq}_{A_1}\subseteq D_j\right)\vee\left(\widetilde{[x_i]}^{S\geq}_{A_2}\subseteq D_j\right)\vee\left(\widetilde{[x_i]}^{H\geq}_{A_3}\subseteq D_j\right)$ **then**
        $\underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)=\underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)\cup\{x_i\}$; $\overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)=\sim\underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(\sim D_j)$;
      end
      if $\left(\widetilde{[x_i]}^{T\geq}_{A_1}\subseteq D_j\right)\wedge\left(\widetilde{[x_i]}^{S\geq}_{A_2}\subseteq D_j\right)\wedge\left(\widetilde{[x_i]}^{H\geq}_{A_3}\subseteq D_j\right)$ **then**
        $\underline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)=\underline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)\cup\{x_i\}$; $\overline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)=\sim\underline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(\sim D_j)$;
      end
    end
    $\rho^{\geq}_{LM}(D_j)=1-\frac{\underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)}{\overline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)}$; $\rho^{\geq}_{TM}(D_j)=1-\frac{\underline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)}{\overline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)}$;
    $\gamma_{LM}(D^+)=\frac{\left|\underline{\widetilde{LM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)\right|}{|U|}$; $\gamma_{TM}(D^+)=\frac{\left|\underline{\widetilde{TM}}^{G_S\geq}_{A_1+A_2+A_3}(D_j)\right|}{|U|}$;
  end
  return: $\rho^{\geq}_{LM}(D_j)$, $\rho^{\geq}_{TM}(D_j)$; $\gamma_{LM}(D^+)$, $\gamma_{TM}(D^+)$.
end

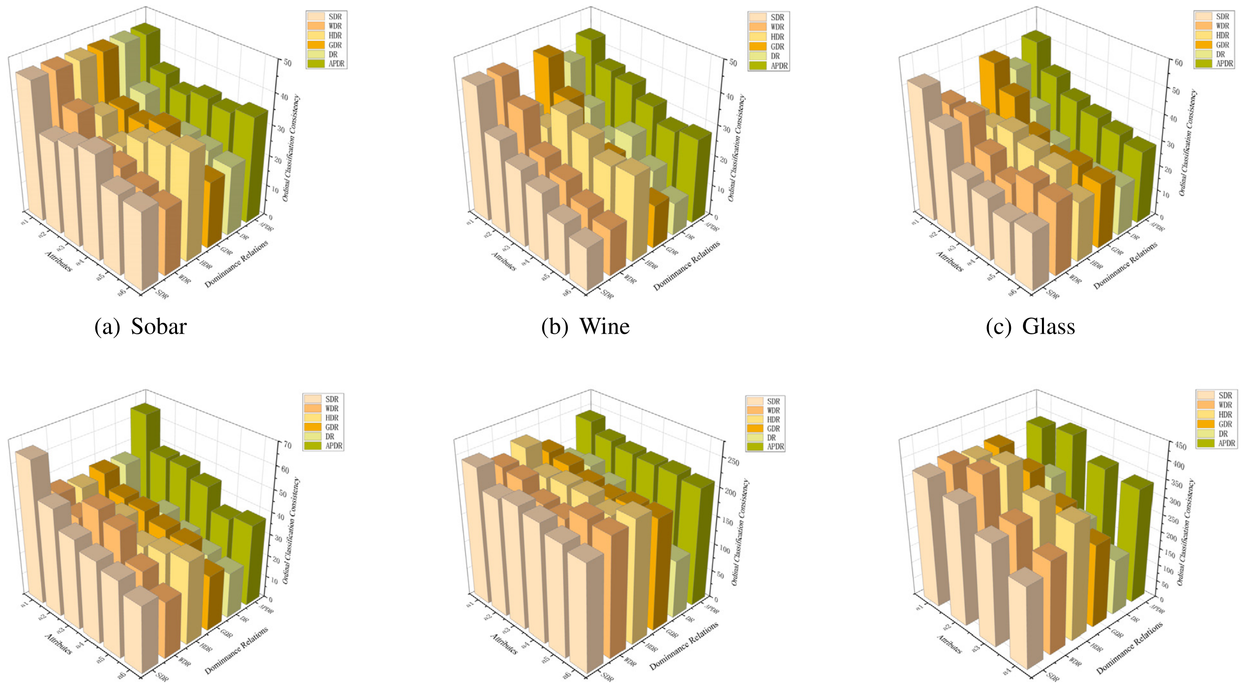
superiority of the SPRS and DPRS will be illustrated based on the eight UCI datasets listed in Table 8. The experiments were executed on a personal computer with the Windows 10 (64-bit) OS, an Intel Core i7 2.5 GHz CPU, and 8.0 GB of RAM. All algorithms were implemented in Python 3.8.

It is important to remember that the foundation of this research is IFODIS. An intuitionistic fuzzy value is composed of a membership degree, non-membership degree, and hesitation degree. Therefore, it is imperative for us to construct intuitionistic fuzzy datasets from the UCI datasets. To ensure feasibility and rationality, we normalized the values in the datasets to construct intuitionistic fuzzy values. In our experiments, the values of the decision attributes in the datasets remained unchanged.

For each dataset used in our experiments, we set the decision classes as the target set. To evaluate the effectiveness of the SPRS, we compared it to a rough set model based on the dominance relation proposed by Xu et al. [24] in Definition 2.1 and the rough set model proposed by Xu et al. [25] in Definition 2.3. Additionally, the rough set model of the strong dominance relation, rough set model of the weak dominance relation, and rough set model of the hesitant dominance relation proposed in this paper were also included in our comparisons.

Table 8
The testing data sets.

Data sets	Data source	Objects	Attributes	Decision class
Sobar-72	UCI	72	19	2
Wine	UCI	178	13	3
Glass	UCI	214	9	6
Wholesale customers	UCI	440	7	3
Indian Liver Patient Dataset (ILPD)	UCI	583	9	2
Banknote authentication	UCI	1372	4	2
Wireless	UCI	2000	7	4
Customer Churn	UCI	3334	10	2



(a) Sobar (b) Wine (c) Glass

(d) Wholesale customers (e) Indian Liver Patient Dataset (ILPD) (f) Banknote authentication

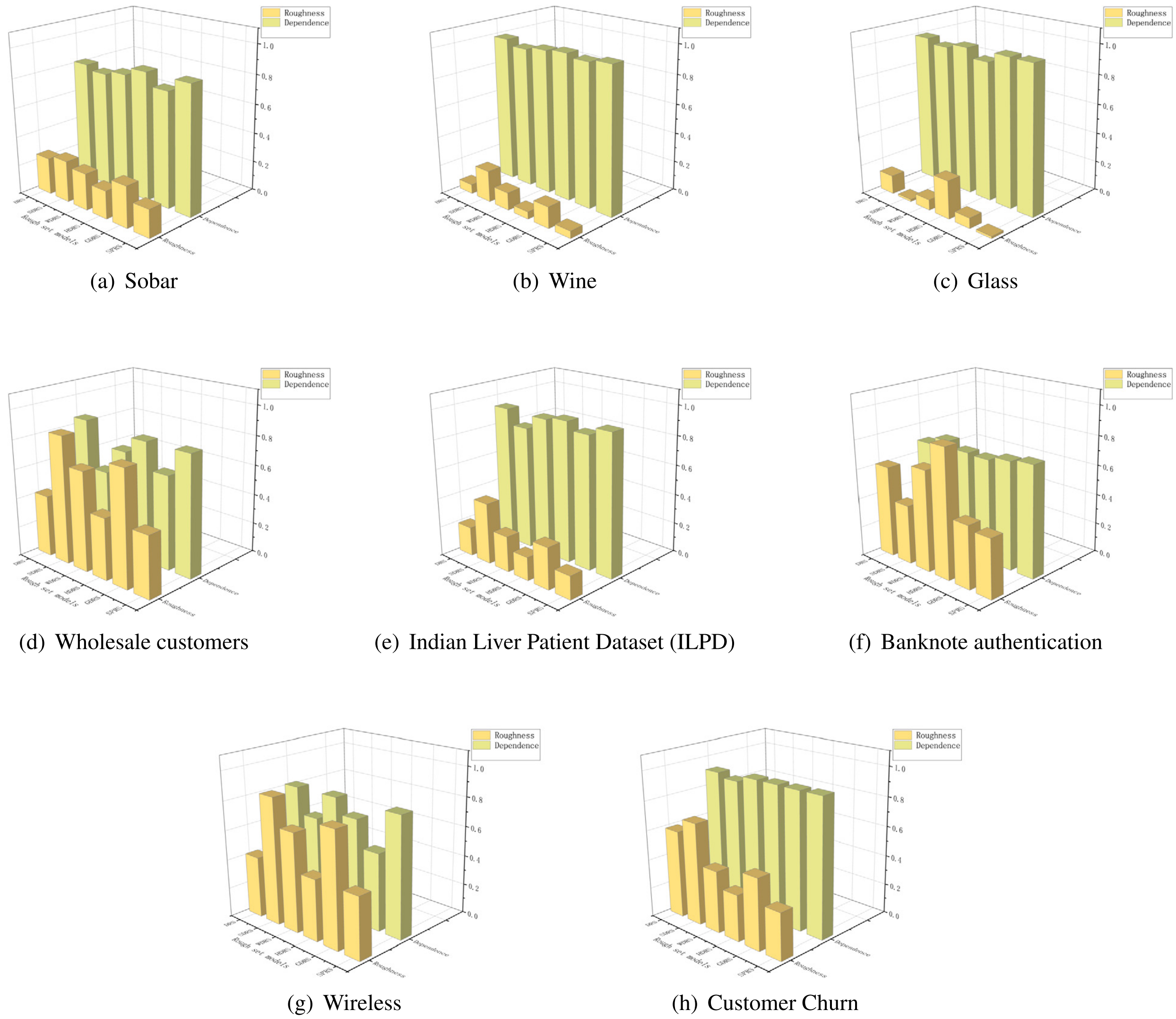
Fig. 6. Columnar charts of the ordinal classification consistency under dominance relations. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Table 9
The roughness and dependence of rough sets under six dominance relations.

Dataset	SDRS		WDRS		HDRS		DRS		GDRS		SPRS	
	$\rho(D_1)$	$\gamma(D^+)$	$\rho(D_1)$	$\gamma(D^+)$	$\rho(D_1)$	$\gamma(D^+)$	$\rho(D_1)$	$\gamma(D^+)$	$\rho(D_1)$	$\gamma(D^+)$	$\rho(D_1)$	$\gamma(D^+)$
Sobar-72	0.2817	0.7849	0.2500	0.8172	**0.1905**	**0.8710**	0.2500	0.8172	0.2817	0.7849	**0.1905**	**0.8710**
Wine	0.2133	0.9507	0.1194	0.9739	**0.0492**	**0.9884**	0.0635	0.9880	0.1449	0.9681	**0.0492**	**0.9884**
Glass	**0.0140**	0.9963	0.0714	0.9926	0.2588	0.9338	0.1267	0.9963	0.0704	**0.9982**	**0.0140**	0.9963
Wholesale customers	0.8624	0.5273	0.6723	0.7087	0.4167	0.8159	**0.4158**	**0.8552**	0.7791	0.6336	0.4167	0.8159
Indian Liver Patient Dataset	0.4125	0.8307	0.2410	0.9247	**0.1579**	**0.9467**	0.1974	0.9333	0.2822	0.8933	**0.1579**	**0.9467**
Banknote authentication	**0.3976**	**0.7462**	0.6790	0.7038	0.8673	0.6958	0.6139	0.6932	0.4248	0.7265	**0.3976**	**0.7462**
Wireless	**0.2714**	**0.9692**	0.4917	0.9034	0.8807	0.7178	0.8645	0.7230	0.6183	0.8726	0.4917	0.9034
Customer Churn	0.6896	0.9347	0.4183	0.9350	**0.3135**	0.9311	0.5924	0.9174	0.4775	**0.9494**	**0.3135**	0.9311

4.1. Ordinal classification consistency of the SPRS

Conditional entropy can measure the consistency of ordinal classification under a certain attribute. Compared to conditional entropy, the core $f_A(x)$ of $\widetilde{RC}^{G\geq}_{d|A}$ varies more significantly and can be represented as



(a) Sobar (b) Wine (c) Glass (d) Wholesale customers (e) Indian Liver Patient Dataset (ILPD) (f) Banknote authentication (g) Wireless (h) Customer Churn

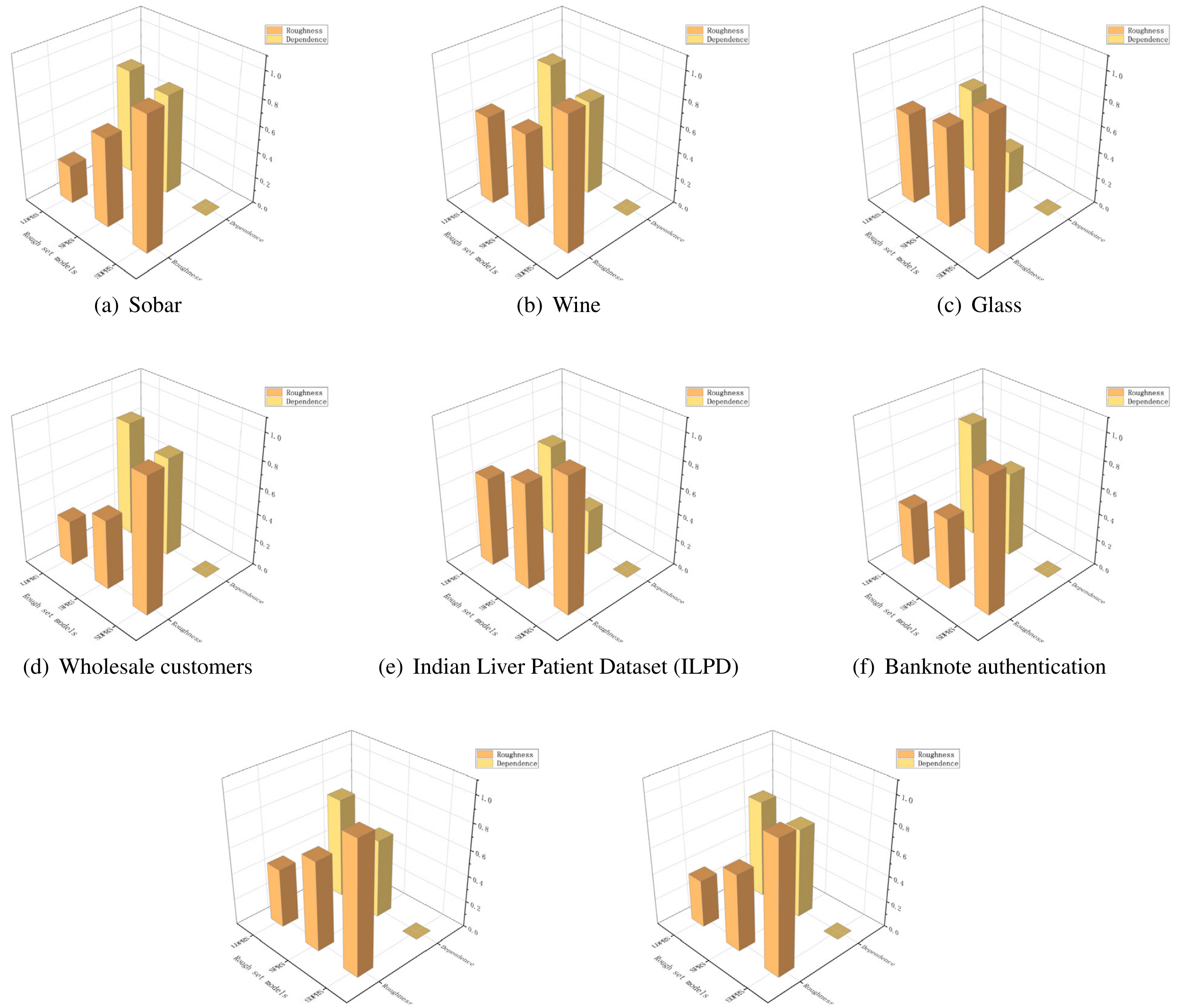
Fig. 7. Columnar charts of roughness and dependence for rough set models under six dominance relations.

Table 10
The roughness and dependence of LDPRS, SPRS and TDPRS.

Dataset	The roughness			The dependence		
	LDPRS	SPRS	TDPRS	LDPRS	SPRS	TDPRS
Sobar-72	0.2917	0.6742	1.0000	0.8056	0.7639	0.0000
Wine	0.6685	0.7043	1.0000	0.8460	0.7134	0.0000
Glass	0.6889	0.7501	1.0000	0.6557	0.3222	0.0000
Wholesale customers	0.3417	0.5247	1.0000	0.8755	0.7528	0.0000
Indian Liver Patient Dataset	0.6690	0.7892	1.0000	0.6964	0.3481	0.0000
Banknote authentication	0.4446	0.5365	1.0000	0.8667	0.6334	0.0000
Wireless	0.4531	0.6820	1.0000	0.7678	0.6026	0.0000
Customer Churn	0.3665	0.5860	1.0000	0.7513	0.6859	0.0000

$$f_A(x) = \sum_{j=1}^{m} \frac{\left|\widetilde{[x_j]}_A^{G\geq} \cap D_d^+(x_j)\right|}{\left|D_d^+(x_j)\right|},$$

where the greater the value of $f_A(x)$, the greater the significance of the dominance relation. Therefore, we use the value of $f_A(x)$ to evaluate the ordinal classification consistency of different dominance relations under each attribute. In our reported results, SDR, WDR, and HDR represent the strong dominance, weak dominance, and hesitant dominance relations,



(a) Sobar (b) Wine (c) Glass

(d) Wholesale customers (e) Indian Liver Patient Dataset (ILPD) (f) Banknote authentication

(g) Wireless (h) Customer Churn

Fig. 8. Columnar charts of roughness and dependence for rough set models of LDPRS, SPRS and TDPRS.

respectively. DR indicates the dominance relation in Definition 2.1 and GDR refers to the general dominance relation in Definition 2.3. Additionally, APDR represents the adjustable-perspective dominance relation proposed in this paper. For simplicity, we randomly selected six datasets and the first six attributes $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ in each dataset as the testing sets. The experimental results are presented in Fig. 6. In Fig. 6, the x axis represents the six selected attributes from each dataset and the y axis represents the values of $f_A(x)$ under the six corresponding dominance relations. Additionally, the z axis represents the ordinal classification consistency of the six dominance relations under different attributes. One can see that the ordinal classification consistency of APDR is significantly higher than that of DR and GDR, and that it maintains a relatively high ordinal classification consistency when compared to SDR, WDR, and HDR. These results demonstrate that APDR is more reliable and effective for ranking datasets from different perspectives.

4.2. Roughness the dependence degree of the SPRS and DPRS

The first experiment presented in this subsection compared the roughness and dependence of the SPRS model proposed in this paper to those of other dominance rough set models. For the rough set models based on the other five dominance relations, we abbreviate the rough set models based on the strong dominance relation, weak dominance relation, and hesitant dominance relation as SDRS, WDRS, and HDRS respectively. Additionally, DRS and GDRS respectively represent rough set models based on Definitions 2.1 and 2.3. It is crucial to consider the selection of target concepts and attributes in the process of conceptual approximation. To facilitate fair comparison, we uniformly consider the first decision class D_1

as the target set. To ensure the reliability of our experiment results, we selected all attributes AT from each dataset during conceptual approximation. The experimental results of Algorithm 1 and Algorithm 2 are presented in Table 9.

In Table 9, among the six dominance rough set models, SPRS maintains a relatively low roughness and high dependency. However, on some datasets, the performance of the SPRS model is slightly inferior to those of some other dominance rough set models. To make comparisons between the SPRS method and other rough set methods more clear and intuitive, we plotted some three-dimensional columnar graphs of the data in Table 9. The results are presented in Fig. 7. One can see that the SPRS is very feasible and effective for conceptual approximation tasks and that it closely matches the requirements of real-world applications and human logical thinking.

Our final experiment compared the roughness and dependence of LDPRS, SPRS, and TDPRS. According to Algorithm 3, we know that the adjustable-perspective dominance relation divides an attribute set into three attribute subsets with different emphases. Consistent with the first experiment, we selected the first decision class as the target concept. The experimental results are presented in Table 10.

According to the calculation results presented in Table 10, the roughness of LDPRS is the smallest, the roughness of TDPRS is the largest, and the roughness of SPRS is between those of LDPRS and TDPRS, which can be proved based on Propositions 3.3 and 3.5. These results conform to the principles of the three models. Similarly, the dependence of LDPRS is the largest, the dependence of TDPRS is the smallest, and the dependence of SPRS is between those of LDPRS and TDPRS. Based on these results, a three-dimensional histogram is presented in Fig. 8.

5. Conclusion

Because classic intuitionistic fuzzy dominance relations do not consider the partial order relations between objects from multiple perspectives, we defined novel dominance relations from three perspectives based on the triangular norms and explored the adjustable-perspective dominance relation according to uncertainty metrics. The adjustable-perspective dominance relation not only reflects the order relations between objects from different perspectives, but also reduces the inconsistency of order classification under different features and decisions. Furthermore, we constructed the SPRS and applied it to several realistic problems. Considering the complexity of practical problems in the real world, we extended the SPRS model to the LDPRS and TDPRS models, and discussed relevant properties and rule extraction. Finally, to verify the effectiveness of the proposed model, we implemented a series of experiments based on eight datasets. The experimental results demonstrated that the proposed model is feasible and effective.

In this paper, only the SPRS and DPRS models were proposed. We verified that the roughness and dependence of these models are significantly improved. However, because the features in information systems are typically multidimensional, how to perform attribute reduction based on three different dominance relations is a point worthy of further study. Additionally, we also want to combine the proposed model with an incremental mechanism to enhance the speed of attribute reduction.

CRediT authorship contribution statement

Xiaoyan Zhang: Conceptualization, Funding acquisition, Investigation, Methodology, Project administration, Supervision, Validation. **Jianglong Hou:** Data curation, Methodology, Software, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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