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A novel concept-cognitive learning method: A perspective from competences

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ABSTRACT

Concept-cognitive learning has been widely used in simulating the human brain to learn concepts. However, the existing concept-cognitive learning models mainly focused on how to acquire knowledge and its properties, but not how to solve the problems. The underlying skills for solving problems are ignored in the cognitive learning process. Indeed, a concept-cognitive learning process is always accompanied with problem solving and skill learning, and the skills are necessary for solving problems. Knowledge space theory is an effective mathematical analysis approach for knowledge assessment. Nevertheless, the existing learning paths for skill were evaluated by constructing the concept lattice, which is a NP hard problem. To overcome these limitations and problems, a novel concept-cognitive learning model from a perspective of competences is proposed. Firstly, knowledge and skills can be represented by item sets and skill sets. And a good semantic explanation between knowledge and skills is provided by a competence-based concept, which presents that one can acquire the most knowledge with the least amount of skills. Secondly, a competence-based concept-cognitive learning model and its properties are put forward to describe the concept-cognitive learning through skills. Moreover, by analyzing the sufficient and necessary relationship between skills and knowledge, a competence-based information granule structure is constructed. Finally, a transformation method of information granules is proposed to convert a general information granule into sufficient and necessary competence-based information granules (i.e., competence-based concepts). And the experimental results of UCI data sets show that the competence-based concept-cognitive learning model is feasible and effective.
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1. Introduction

Concept Learning (CL) [1–6] is a branch of the artificial intelligence to simulate the human brain through a computer system. By learning the concepts, one can gradually obtain the knowledge and its properties. Concept-cognitive learning (CCL) always starts from some inaccurate, uncertain and partially real problems, and finally achieves varying degrees of cognitive learning through the self-improving in the cognitive system. CCL has been discussed from granular computing (GrC) [7–9] to simplify the problems. By describing proper information granules and dividing complex structures into simpler ones, granular computing can simulate human thinking patterns, analyze and solve many practical problems [2–11]. By using basic information granules as basic information processing units, GrC can effectively reduce the computational complexity. Zadeh [7] firstly proposed and discussed the fuzzy information granulation. Yao [8] discussed integrative levels of granularity, granular computing on basic issues and possible solution. Wu et al. [9] explored knowledge reduction with granular computing. Recently, many investigators study the concept-cognitive learning model (CCLM) from the view of granular computing [2–5,10–14].

Concept-cognitive learning model is related to types of concepts [4]. Various concepts, for instance, formal concept [15], attribute-oriented concept [16], object-oriented concept [17], L-fuzzy concept [18] and three-way concept [19], carry certain meanings, and provide different semantic interpretation. Concept-cognitive learning models (CCLMs) can satisfy different demands by using different generation mechanisms of concepts. Zhang et al. [2] studied a novel framework of concept cognitive model with granule computing. Xu et al. [3] used information granules to describe the sufficient and necessary relationship between attributes and objects, and presented a transformation method which converted an arbitrary information granule into sufficient

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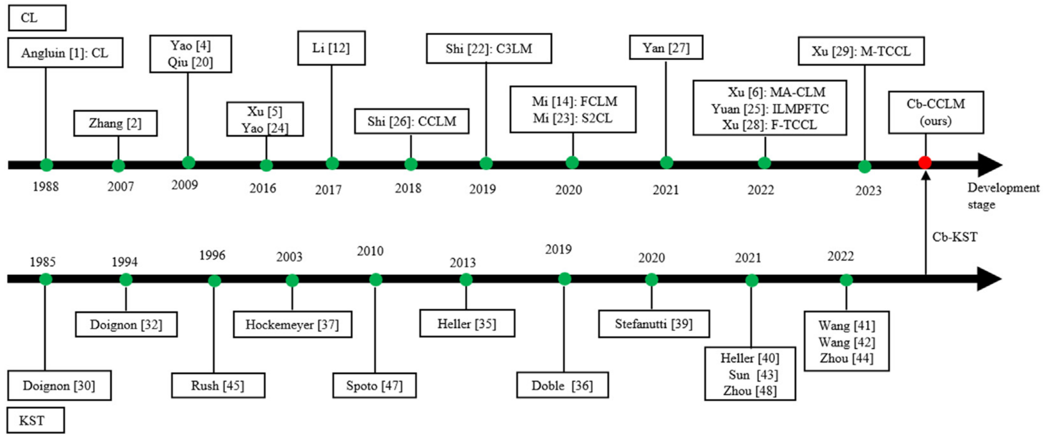

Fig. 1. The development stage of CL and KST.

and necessary information granules. And the approach of Ref. [3] was extended to fuzzy formal context [5]. Yao [4] presented a concept learning model from cognitive informatics and granular computing. Qiu et al. [20] proposed a granular computing system model, which used concept granular iterative operator to form concepts. Li et al. [10,11] used cognitive granular concepts (i.e., object granular concepts and attribute granular concepts) to calculate approximate cognitive concepts according to cues. Zhao et al. [21] used the information similarity matching method to obtain the optimal approximate cognitive concept under the maximum matching principle. Xu [6], Shi [22] and Mi [14,23] presented some CCLMs, which aimed at solving classification tasks and acquire the conceptual generalization capability. These different cognitive models have different cognitive mechanisms such as granular concept cognitive system [2–5,13,20], three-way concept-cognitive learning [12,24], approximate CCL [10,11], conceptual clustering [6,14,22,23], incremental learning [25–27] and dynamic concept learning [28,29]. These CCLMs mainly focused on how to acquire knowledge and its commonalities, but neglected how to solve the problems. In fact, rather than acquiring knowledge, the skills learned for solving problems are the key to the solution of the problems.

Knowledge space theory (KST) proposed by Doignon [30] is an effective mathematical analysis approach for knowledge assessment and adaptive evaluation [31–38]. The extensions of KST are competence-based knowledge space theory (Cb-KST) [31–35], the polytomous generalization of knowledge space theory [39–42] and fuzzy knowledge space theory [43,44]. The development stage of the CL and KST is shown in Fig. 1.

Cb-KST has been used for skill evaluation [35,36] by computer systems, such as APeLS system [37] and the iClass system [38]. Rusch et al. [45] connected KST with FCA. Xu [46] and Spoto [47] discussed the skill reduction. Zhou [48] obtained the skill assessment and the learning paths of skill from the attribute-oriented formal concept lattice in a skill context. Sun [43] and Zhou [44] aimed at constructing the fuzzy knowledge structures delineated by fuzzy skill maps. Cb-KST was studied through a skill map or skill multimap [32–35]. There are two special skill maps: conjunctive and disjunctive model [32,35]. A disjunctive model of skill map allots a singleton skill to each item. For example, if one wants to calculate the value of the item q_1 : 16+16+16, one only needs to master any one skill of the skill set {*addition*, *multiplication*}. A conjunctive model of skill map assigns a nonempty subset of skills to each item. For instance, one can correctly calculate the value of the item q_2 : $\frac{48}{3}+5$, if one masters all skills of the skill set {*addition*, *division*}. A skill multimap may allot multiple solutions to solve one item, and any method may have more than one skill. Such as, one wants to calculate the value of the item q_3 : $1+2+\cdots+100$, one only needs to master one solution (i.e., a skill set) of $\{\{addition\}, \{summation\ formula,\ multiplication, division\}\}$. Therefore, if one only learns the properties and rules of the arithmetic operation, but not the calculation skills of {*addition*, *subtraction*, *multiplication*, *division*}, one may fail to solve $\{q_1, q_2, q_3\}$. Thus, the skill learning is necessary for solving problems.

For the item q_2 : $\frac{48}{3}+5$, one needs to master the minimal skill set {*addition*, *division*} for solving it. In additional, the skill set {*addition*} is needed, but not enough for solving q_2. Thus, the skill *addition* is a sufficient skill for solving q_2. The skill set {*addition*, *multiplication*, *division*} is too much for solving q_2, where the skill *multiplication* is not needed. Then {*addition*, *multiplication*, *division*} is a necessary skill set for solving q_2. The skill *subtraction* is not able to solve q_2. Then the skill set {*subtraction*} is neither sufficient nor necessary for solving q_2. Therefore, there is a sufficient and necessary relationship between skills and knowledge. In general, one starts to learn and master simple skills to solve simple problems. Then one gradually learns some complex skills to deal with complex problems. That is, as skills are learned and mastered, more knowledge will be obtained and more problems will be solved.

The motivations of this article are described from the two perspectives as follows.

(1) The existing CCLMs [3,5,10,12] mainly aimed at how to learn the knowledge and its properties. However, they cannot deal with how to solve the problems, and what skills are involved to the solution of the problems. Therefore, it limits the expansion and application of CCLM. Thus, we explore a novel CCLM from the view of competences.

(2) As mentioned above, learning paths for skills [48] were evaluated from the whole concept lattice. And the method of [48] was extended to fuzzy knowledge space [44]. Therefore, the computational complexity of [44,48] is expensive. In fact, it is a NP hard problem. In addition, by using all granular concepts to calculate cognitive concepts from each cue, the cognitive concept learning via granular computing [10] has reduced the computational complexity. However, the running times are still a little higher in big data. Thus, to reduce the computational complexity, we attempt to construct an information granule structure from the relationship between knowledge and skills.

Up to now, few studies discuss CCLM from the perspective of competences. To overcome these limitations and problems, a novel CCLM is investigated with skill learning in this article. The main contributions of this paper are described as follow.

(1) Based on FCA, KST and GrC, a competence-based concept-cognitive learning model (Cb-CCLM) is proposed to connect

knowledge with skills, and its basic theories and properties are discussed.

(2) Specially, in a skill context, a competence-based concept (Cb-concept) can character a good semantic interpretation between knowledge and skills. If one acquires a Cb-concept, then one can obtain the most knowledge with the least amount of skills. It conforms to the cognitive learning expectations from the view of economic costs in real life. In this case, the skills are called necessary and sufficient for the knowledge. Otherwise, the skills may be necessary, sufficient, or neither necessary nor sufficient for the knowledge. To describe the necessary and sufficient relationship between knowledge and skills, we propose a granule description which is an extension of [3], and explore the theories and properties of the information granule structure.

(3) Essentially, the concept-cognitive learning processes of the Cb-CCLM are the changes between the item sets and the skill sets. Then, without constructing the concept lattice or all granular concepts, we present a transformation method between information granules. And we can construct sufficient and necessary competence-based information granules (i.e., Cb-concepts) from a general information granule. Moreover, the obtained Cb-concepts can guide one to achieve the personalized learning.

(4) Algorithms of the transformation between information granules are presented in this paper. And the experimental results show that Algorithm 1 in the Cb-CCLM is feasible and effective even in big data.

The rest of this paper will be organized as follows: A brief overview of FCA and KST is described in Section 2. A Cb-CCLM is explored and its basic properties are discussed in Section 3. In Section 4, an information granule structure and its theories are presented. A transformation between information granules and algorithms of the transformation are proposed. In Section 5, the experiments of Algorithm 1 are tested in some UCI data sets. The conclusions are summarized in Section 6.

2. FCA and KST: a brief overview

2.1. Formal concept analysis theory

The basic concepts and conclusions of FCA [9,16] are introduced as follows.

Definition 2.1 (*[16]*). A formal context is a triple (I, T, R), where $I = \{i_1, i_2, \ldots, i_n\}$ and $T = \{t_1, t_2, \ldots, t_m\}$ are two nonempty finite sets of objects and attributes, respectively. R is a binary relation between I and T. When $(p, s) \in R$, the attribute s is possessed by the object p. When $(p, s) \notin R$, the object p does not possess the attribute s. In fact, the formal context is usually represented by a table containing values 1 and 0, where 1 means the row object possesses the column attribute, and 0 means the row object does not possess the column attribute.

Definition 2.2 (*[16]*). Let (I, T, R) be a formal context. For $p \in I$, $s \in T$, $P \subseteq I$, and $S \subseteq T$, we define the four operators as follows:

(1) $p^* = \{s \in T \mid (p, s) \in R\}$.
(2) $s^* = \{p \in I \mid (p, s) \in R\}$.
(3) $P^\diamond = \{s \in T \mid s^* \cap P \neq \emptyset\}$.
(4) $S^\square = \{p \in I \mid p^* \subseteq S\}$.

Where, p^* is the set of attributes owned by the object p, s^* is the set of objects which possess the attribute s, $P^\diamond$ represents the set of attributes possessed by the objects in P, and $S^\square$ represents the set of objects whose attributes are all in S. In addition, for any $p \in I$, $\{p\}^\diamond$ is denoted as $p^\diamond$ for short. Similarly, for any $s \in T$, denote $s^\square$ for convenience instead of $\{s\}^\square$.

The operators "$\diamond$" and "$\square$" have the fundamental properties.

Lemma 2.1 (*[16]*). *Let (I, T, R) be a formal context. For $P, P_1, P_2 \subseteq I$ and $S, S_1, S_2 \subseteq T$:*

(1) $P_1 \subseteq P_2 \Rightarrow P_1^\diamond \subseteq P_2^\diamond$, $S_1 \subseteq S_2 \Rightarrow S_1^\square \subseteq S_2^\square$.
(2) $P \subseteq P^{\diamond\square}$, $S^{\square\diamond} \subseteq S$.
(3) $P^\diamond = P^{\diamond\square\diamond}$, $S^\square = S^{\square\diamond\square}$.
(4) $(P_1 \cup P_2)^\diamond = P_1^\diamond \cup P_2^\diamond$, $(S_1 \cap S_2)^\square = S_1^\square \cap S_2^\square$.

Lemma 2.2 (*[49]*). *Let (I, T, R) be a formal context. For $P \subseteq I$ and $S \subseteq T$, then $P \subseteq S^\square$ if and only if $P^\diamond \subseteq S$.*

Definition 2.3 (*[16]*). Let (I, T, R) be a formal context. For $P \subseteq I$, $S \subseteq T$, if $P^\diamond = S$ and $S^\square = P$, then the pair (P, S) is an attribute-oriented formal concept, where P is the extension and S is the intension of (P, S).

The family of all attribute-oriented formal concepts is denoted as $L_T(I, T, R) = \{(P, S) \mid P^\diamond = S, S^\square = P\}$. For $(P_1, S_1), (P_2, S_2) \in L_T(I, T, R)$, "$\leqslant$" is ordered by $(P_1, S_1) \leqslant (P_2, S_2) \Leftrightarrow P_1 \subseteq P_2$ $(S_1 \subseteq S_2)$, the operators "$\vee_T$" and "$\wedge_T$" are defined by:

$(P_1, S_1) \wedge_T (P_2, S_2) = (P_1 \bigcap P_2, (S_1 \bigcap S_2)^{\square\diamond})$,
$(P_1, S_1) \vee_T (P_2, S_2) = ((P_1 \bigcup P_2)^{\diamond\square}, S_1 \bigcup S_2)$.

Then $L_T(I, T, R)$ is a complete lattice, which is an attribute-oriented formal concept lattice. For $P \subseteq I$ and $S \subseteq T$, $(P^{\diamond\square}, P^\diamond)$ and $(S^\square, S^{\square\diamond})$ are attribute-oriented formal concepts.

2.2. Knowledge space theory

In this paper, the power set of a nonempty finite set U is denoted as 2^U. Let $I = \{i_1, i_2, \ldots, i_n\}$ and $T = \{t_1, t_2, \ldots, t_m\}$ be two nonempty finite sets of items and skills, and the skills of T are relevant to solve the items of I. The items and skills can be connected by a skill map or skill multimap.

Definition 2.4 (*[31]*). A skill map is a triple (I, T, τ), where I is a nonempty set of items, T is a nonempty set of skills, and τ is a mapping from I to $2^T \setminus \{\emptyset\}$. We sometimes refer to the function τ as the skill map. For any $p \in I$, the subset $\tau(p) \subseteq T$ is the set of skills assigned to p (by the skill map τ).

Let (I, T, τ) be a skill map and $S \subseteq T$, we denote that $P(S) \subseteq I$ is the knowledge state delineated by S via the conjunctive model if $P(S) = \{p \in I \mid \tau(p) \subseteq S\}$. The knowledge state $P(S) \subseteq I$ delineated by S via the disjunctive model is specified by $P(S) = \{p \in I \mid \tau(p) \bigcap S \neq \emptyset\}$. Through all subsets $S \subseteq T$, the collection of all knowledge states is $\mathcal{P}$, and $(I, \mathcal{P})$ is called a knowledge structure induced by τ, where $\mathcal{P}$ includes $\emptyset$ and I. Usually $\mathcal{P}$ is called a knowledge structure.

We consider an example with $I = \{1, 2, 3\}$ and $T = \{s, t, u\}$. Let $\tau : I \rightarrow 2^T \setminus \{\emptyset\}$ be defined by $\tau(1) = \{s, t\}$, $\tau(2) = \{t\}$, $\tau(3) = \{u\}$. Then, τ is a skill map. For $S = \{t\}$, then the knowledge state delineated by S via the conjunctive model is $\{2\}$. The knowledge structure delineated by the conjunctive model is $\{\emptyset, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. For the other case, $S = \{t\}$ can delineate the knowledge state $\{1, 2\}$ via the disjunctive model. The knowledge structure delineated by the disjunctive model is $\{\emptyset, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$.

Remark 2.1 (*[35]*). The knowledge structure delineated by the conjunctive model of skill map is a simple closure space. And the knowledge structures delineated via the disjunctive and the conjunctive model by the same skill map are dual one to the other. Thus, the knowledge structure delineated via the disjunctive model of skill map is a knowledge space. In this paper, we mainly discuss the conjunctive model of skill map.

Definition 2.5 (*[33,35]*). A skill multimap is a triple (I, T, υ), where : $\upsilon : I \rightarrow (2^{2^T \setminus \{\emptyset\}}) \setminus \{\emptyset\}$. For any $p \in I$, the mapping υ satisfies:

(1) $\upsilon(p) \neq \emptyset$,
(2) $S \neq \emptyset$, for any $S \in \upsilon(p)$.

The $S \in \upsilon(p)$ is called a competency. Additionally, if the competencies in each $\upsilon(p)$ are pairwise incomparable (with respect to set inclusion), then (I, T, υ) is called a skill function [35]. That is, if $S_1, S_2 \in \upsilon(p)$, then S_1, S_2 are incomparable with respect to set inclusion of set.

A skill function may assign more than one competency to an item, and the assigned competencies should even be minimally sufficient for solving the item. Minimality means that competencies assigned to an item are pairwise incomparable, and it is a property called incomparability condition. We usually use υ to represent the skill multimap or skill function.

We consider an example with $I = \{1, 2, 3\}$ and $T = \{s, t, u\}$. Let the skill multimap υ and $\widetilde{\upsilon}$ be defined by

$\upsilon(1) = \{\{s, t\}, \{t, u\}\}$, $\upsilon(2) = \{\{t\}, \{t, u\}\}$, $\upsilon(3) = \{\{u\}\}$.
$\widetilde{\upsilon}(1) = \{\{s, t\}, \{t, u\}\}$, $\widetilde{\upsilon}(2) = \{\{t\}\}$, $\widetilde{\upsilon}(3) = \{\{u\}\}$.

Note that the skill multimap υ does not satisfy the incomparability condition as the two competencies in $\upsilon(2)$ are nested. The skill multimap $\widetilde{\upsilon}$ is a skill function, and $\widetilde{\upsilon}$ is called a reduction of υ [35]. We usually use the skill multimap with incomparability condition. The detailed description and background of KST can be seen in Refs. [31,33,35].

Therefore, for any $p \in I, S \in \upsilon(p)$ is a minimal subset of T to solve it. Then, one needs to master at least all the skills of one competency S to solve p. For $S \subseteq T$, in ideal conditions (i.e., there are no careless errors or lucky guesses), we denote that $P(S) = \{p \in I \mid \exists C \in \upsilon(p), C \subseteq S\}$, then $P(S)$ is a knowledge state induced by S via υ. Through all subsets $S \subseteq T$, the collection of all knowledge states is $\mathcal{P}$, and $(I, \mathcal{P})$ is called a knowledge structure induced by υ, where $\mathcal{P}$ includes $\emptyset$ and I. Usually $\mathcal{P}$ is called a knowledge structure.

The skill subset $S \subseteq T$ that one has mastered is defined as a competence state. Let $\mathcal{T}$ be the collection of all competence states. $(T, \mathcal{T})$ is called as a competence structure, where $\mathcal{T}$ includes $\emptyset$ and T. In this paper, the power set 2^T is considered as a competence structure. Then, S $(S \subseteq T)$ is a competence state. In Cb-KST, knowledge and skills can be reflected by knowledge state and competence state.

Example 2.1. Let (I_1, T_1, υ) be a skill multimap, where $I_1 = \{1, 2, 3, 4\}$, $T_1 = \{t, s, r, \upsilon\}$, $\upsilon(1) = \{\{s, r\}, \{r, \upsilon\}\}$, $\upsilon(2) = \{\{t, s\}, \{t, r\}\}$, $\upsilon(3) = \{\{r\}\}$, $\upsilon(4) = \{\{t\}\}$. Let $S = \{s, r\}$ be a competence state, and $P(S) = \{p \in I_1 \mid \exists C \in \upsilon(p), C \subseteq S\}$ be a knowledge state induced by S via υ. For $\{s, r\} \in \upsilon(1)$, and $\{s, r\} \subseteq S$, then $1 \in P(S)$. For $\{r\} \in \upsilon(3)$, and $\{r\} \subseteq S$, then $3 \in P(S)$. Therefore, $P(S) = \{1, 3\}$ is the knowledge state induced by S. Through all subsets $S \subseteq T_1$, the knowledge structure induced by υ is $\mathcal{P}_1 = \{\emptyset, \{3\}, \{4\}, \{1, 3\}, \{2, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$.

2.2.1. Knowledge structure induced by the conjunctive model of skill map

Let (I, T, τ) be the conjunctive model of skill map. For $p \in I$, $\emptyset \subset C \subseteq T$, and $\tau(p) = C$, then $s \in C$ if and only if $(p, s) \in R$, the conjunctive model of skill map can be converted to a formal context (I, T, R), which is named a skill context. In a skill context (I, T, R), the attribute-oriented formal concept lattice formed by the operators "$\diamond$" and "$\square$" of Definition 2.2 is $L_T(I, T, R)$. The pair $(P, S) \in L_T(I, T, R)$ is called a competence-based concept (Cb-concept), which is combined by knowledge state P and competence state S. Knowledge state P is induced by competence state S via τ. In a skill context (I, T, R), for any $p \in I$ and $\tau(p) = C$, there is $p^{\diamond} = C$.

Lemma 2.3 ([48]). *Let (I, T, τ) be the conjunctive model of skill map, and (I, T, R) is a skill context corresponding with (I, T, τ). For any $(P, S) \in L_T(I, T, R)$, P is the knowledge state induced by S via τ, then $\mathcal{P} = \{P \mid (P, S) \in L_T(I, T, R)\}$ is a simple closure space.*

Table 1
The skill context (I_2, T_2, R).

$I_2 \backslash T_2$	t	s	r	υ	w
1	0	1	1	1	1
2	0	1	1	0	0
3	1	0	0	0	1
4	0	1	0	0	0
5	1	1	0	0	1

Lemma 2.4 ([48]). *Let (I, T, τ) be the conjunctive model of skill map, and (I, T, R) is a skill context corresponding with (I, T, τ). For any $(P, S) \in L_T(I, T, R)$, S is the minimal competence state which can induce the knowledge state P.*

Proposition 2.1. *Let (I, T, τ) be the conjunctive model of skill map, and (I, T, R) is a skill context corresponding with (I, T, τ). For any $(P, S) \in L_T(I, T, R)$, P is the maximal knowledge state induced by the competence state S.*

Proof. Let $P \subset P' \subseteq I$, and P, P' are knowledge states induced by S. Then there is a $p \in P'$ and $p \notin P$, which makes $C \subseteq S$, where $\emptyset \subset \tau(p) = C \subseteq T$. Then $p \in P(S) = \{p \in I \mid \tau(p) = C \subseteq S\}$, it is a contradiction with $p \notin P$. Thus, P is the maximal knowledge state induced by competence state S.

Therefore, Proposition 2.1 can be proven from the perspective of the generation of knowledge states. By combining Lemma 2.4 and Proposition 2.1, if one acquires a Cb-concept, then one can solve the most items with the least amount of skills.

Example 2.2. Let (I_2, T_2, τ) be the conjunctive model of skill map, where $I_2 = \{1, 2, 3, 4, 5\}$, $T_2 = \{t, s, r, \upsilon, w\}$, $\tau(1) = \{s, r, \upsilon, w\}$, $\tau(2) = \{s, r\}$, $\tau(3) = \{t, w\}$, $\tau(4) = \{s\}$, $\tau(5) = \{t, s, w\}$. The skill context (I_2, T_2, R) corresponding with (I_2, T_2, τ) is in Table 1.

Thus, the attribute-oriented formal concept lattice of the skill context (I_2, T_2, R) is:

$L_T(I_2, T_2, R) = \{(\emptyset, \emptyset), (4, s), (3, tw), (24, sr), (345, stw), (124, sr\upsilon w), (2345, strw), (12345, tsr\upsilon w)\}$. For a Cb-concept $(24, sr)$, 24 represents the knowledge state $P = \{2, 4\}$, sr means that the competence state is $S = \{s, r\}$. The knowledge structure induced by τ is $\mathcal{P}_2 = \{\emptyset, \{3\}, \{4\}, \{2, 4\}, \{3, 4, 5\}, \{1, 2, 4\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}$.

2.2.2. Knowledge structure induced by a skill multimap

Let (I, T, υ) be a skill multimap, and $r = \prod_{p \in I} |\upsilon(p)|$. When $r > 1$, the skill multimap can be decomposed into r conjunctive model of skill maps, i.e., the skill multimap (I, T, υ) can be decomposed into r skill contexts.

Definition 2.6 ([48]). Let (I, T, υ) be a skill multimap, and $r = \prod_{p \in I} |\upsilon(p)|$. Then the skill multimap (I, T, υ) can be decomposed into skill contexts (I, T, R_k) $(k = 1, 2, \dots, r)$. Let

(D1) $L(I, T, \upsilon) = \bigcup_{k=1}^{r} L_T(I, T, R_k)$.

(D2) $L_\upsilon(I, T, \upsilon) = \{(\bigcup_{k=1}^{m} P_i, S) \mid (P_i, S_i) \in L(I, T, \upsilon), S = S_1 = \cdots = S_m, 1 \leqslant m \leqslant r\}$.

Lemma 2.5 ([48]). *Let (I, T, υ) be a skill multimap. For any $(P, S) \in L_\upsilon(I, T, \upsilon)$, the knowledge state P is induced by the competence state S via υ, then $\mathcal{P} = \{P \mid (P, S) \in L_\upsilon(I, T, \upsilon)\}$ is a knowledge structure induced by υ.*

Example 2.3. In Example 2.1, for the skill multimap (I_1, T_1, υ), and $r = \prod_{p \in I} |\upsilon(p)| = 4$. Then the skill multimap (I_1, T_1, υ) can be decomposed into 4 conjunctive model of skill maps, which are listed as follows:

(1) The skill map (I_1, T_1, τ_1), where $\tau_1(1) = \{s, r\}$, $\tau_1(2) = \{t, s\}$, $\tau_1(3) = \{r\}$, $\tau_1(4) = \{t\}$.

(2) The skill map (I_1, T_1, τ_2), where $\tau_2(1) = \{s, r\}$, $\tau_2(2) = \{t, r\}$, $\tau_2(3) = \{r\}$, $\tau_2(4) = \{t\}$.

(3) The skill map (I_1, T_1, τ_3), where $\tau_3(1) = \{r, v\}$, $\tau_3(2) = \{t, s\}$, $\tau_3(3) = \{r\}$, $\tau_3(4) = \{t\}$.

(4) The skill map (I_1, T_1, τ_4), where $\tau_4(1) = \{r, v\}$, $\tau_4(2) = \{t, r\}$, $\tau_4(3) = \{r\}$, $\tau_4(4) = \{t\}$.

Therefore, the skill multimap (I_1, T_1, υ) can be converted to four skill contexts. Then by Definition 2.6 and Lemma 2.5 : $L(I_1, T_1, \upsilon) = \{(\emptyset, \emptyset), (3, r), (4, t), (24, st), (13, sr), (13, r\upsilon), (34, tr), (234, tr), (234, str), (134, tr\upsilon), (1234, str), (1234, tr\upsilon), (1234, tsr\upsilon)\}$.

$L_\upsilon(I_1, T_1, \upsilon) = \{(\emptyset, \emptyset), (3, r), (4, t), (24, st), (13, sr), (13, r\upsilon), (234, tr), (1234, str), (1234, tr\upsilon), (1234, tsr\upsilon)\}$. and the knowledge structure induced by (I_1, T_1, υ) is $\mathcal{P}_3 = \{\emptyset, \{3\}, \{4\}, \{1, 3\}, \{2, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$. And it is consistent with Example 2.1.

3. A competence-based concept-cognitive learning model (Cb-CCLM)

One needs skills such as calculation, reasoning and imagination thinking, to solve some specific problems. Therefore, the concept-cognitive learning process is always companied with skill learning. One can obtain the knowledge state based on the responses to some items. Only when one learns and gets the effective skills [50], the knowledge state will change. From Lemma 2.4 and Proposition 2.1, when the skill set and the item set form a Cb-concept in a skill context, one can acquire the most items with the least amount of skills. In this case, the skills are necessary and sufficient for the knowledge from the view of a Cb-concept. Then, one masters certain skills or laws of things (i.e., Cb-concepts). When the skill set and the item set cannot form a Cb-concept, the skills may be too much, too little, or unable to solve the items. Therefore, the skills may be necessary, sufficient, or neither necessary nor sufficient for the knowledge. As analyzed above, there is a sufficient and necessary relationship between skills and knowledge.

In general, the concept-cognitive learning process starts with unknown knowledge or skills, or inconsistent information. One may learn and master some simple skills, and gradually improve the skills. Consequently, one can obtain the sufficient or necessary skills for the knowledge little by little, and finally master the sufficient and necessary skills for solving the problems. As skills are learned and mastered, more knowledge will be obtained and more problems will be solved. Intrinsically, a cognitive learning process based on a skill context is the changes between the item sets and the skill sets. In the other words, the concept-cognitive learning process needs to consider the sufficient and necessary relationship between the item sets and the skill sets, and use the transformation between them to judge and solve problems. Therefore, two operators between knowledge and skills are proposed to construct a new concept-cognitive learning model with skills.

Let I and T be two nonempty finite sets of items and skills, respectively. Denote $L_1 = 2^I$ and $L_2 = 2^T$, respectively. For $p_1, p_2 \in L_1$ and $s_1, s_2 \in L_2$, we define " $\leqslant$ ", "$\wedge$" and "$\vee$" as follows:

$p_1 \leqslant p_2$ $(s_1 \leqslant s_2) \Leftrightarrow p_1 \subseteq p_2$ $(s_1 \subseteq s_2)$,

$p_1 \wedge p_2$ $(s_1 \wedge s_2) \Leftrightarrow p_1 \cap p_2$ $(s_1 \cap s_2)$,

$p_1 \vee p_2$ $(s_1 \vee s_2) \Leftrightarrow p_1 \cup p_2$ $(s_1 \cup s_2)$.

It is easy to verify that $(L_1, \leqslant)$ and $(L_2, \leqslant)$ are complete lattices. For convenience, denote L_1 and L_2 as two complete lattices of items and skills, respectively. Denote 0_L and 1_L as the bottom element and the top element of complete lattice L.

Definition 3.1. Let L_1 and L_2 be two nonempty finite complete lattices of items and skills, respectively. For $p_1, p_2 \in L_1$ and $s_1, s_2 \in L_2$, $F : L_1 \to L_2$ and $G : L_2 \to L_1$ are two dual competence-based cognitive learning operators, if F and G satisfy:

(1) $F(0_{L_1}) = 0_{L_2}$, $F(1_{L_1}) = 1_{L_2}$.

(2) $F(p_1 \vee p_2) = F(p_1) \vee F(p_2)$.

(3) $G(0_{L_2}) = 0_{L_1}$, $G(1_{L_2}) = 1_{L_1}$.

(4) $G(s_1 \wedge s_2) = G(s_1) \wedge G(s_2)$.

Definition 3.2. A quadruple (L_1, L_2, F, G) is a competence-based concept-cognitive learning model (Cb-CCLM), if F and G are two dual competence-based cognitive learning operators, and satisfy:

(1) $G \circ F(p) \geqslant p$,

(2) $F \circ G(s) \leqslant s$, where $G \circ F(p)$ and $F \circ G(s)$ represent $G(F(p))$ and $F(G(s))$, respectively.

The two operators F and G can characterize the changes between the knowledge and skills in the cognitive process of the Cb-CCLM.

Remark 3.1. (1) For $p \in L_1$, then $F(p) \in L_2$ represents the minimal skill set related to solve the items of p. For $s \in L_2$, then $G(s) \in L_1$ contains the items which can be solved by s. (2) In fact, $0_{L_1} = \emptyset$, $1_{L_1} = I$, $0_{L_2} = \emptyset$ and $1_{L_2} = T$. Obviously, if $p = 0_{L_1}$, i.e., $p = \emptyset$, then one does not master any skills of T, and $F(p) = \emptyset$. Thus, $F(0_{L_1}) = 0_{L_2}$. If $p = 1_{L_1}$, i.e., $p = I$, then one must master all skills of T, and $F(p) = T$. Thus, $F(1_{L_1}) = 1_{L_2}$. On the other hand, if $s = 0_{L_2}$, i.e., $s = \emptyset$, then one can not solve any items of I, and $G(s) = \emptyset$. Thus, $G(0_{L_2}) = 0_{L_1}$. If $s = 1_{L_2}$, i.e., $s = T$, then one can solve all items of I, and $G(s) = I$. Thus, $G(1_{L_2}) = 1_{L_1}$. (3) For any $p_1, p_2 \in L_1$, then $p_1 \vee p_2 \in L_1$. The skills which solve the items of $p_1 \vee p_2$ can solve the items of p_1 or p_2. Thus, $F(p_1 \vee p_2)$ is in $F(p_1)$ or $F(p_2)$, then $F(p_1 \vee p_2) \leqslant F(p_1) \vee F(p_2)$. On the other hand, the skills of $F(p_1)$ and $F(p_2)$ can solve the items of p_1 and p_2 respectively. Then, $F(p_1)$ and $F(p_2)$ can solve the items of p_1 or p_2, i.e., $F(p_1) \vee F(p_2) \leqslant F(p_1 \vee p_2)$. Thus, $F(p_1 \vee p_2) = F(p_1) \vee F(p_2)$. It means that the more items require more skills for solving them. Or when two individuals learn together, their skills will enhance. For $s_1, s_2 \in L_2$, then $s_1 \wedge s_2 \in L_2$. The items solved by skills of $s_1 \wedge s_2$ are clearly in $G(s_1) \wedge G(s_2)$. Moreover, the items solved by s_1 and s_2 are in $G(s_1) \wedge G(s_2)$, and the skills of $s_1 \wedge s_2$ can solve the items of $G(s_1) \wedge G(s_2)$. Then $G(s_1 \wedge s_2) = G(s_1) \wedge G(s_2)$. Thus, the operators F and G of Cb-CCLM can reflect the relationship between knowledge and skills from the view of competences.

The two operators F and G have some important properties.

Proposition 3.1. *Let (L_1, L_2, F, G) be a Cb-CCLM. For any $p, p_1, p_2 \in L_1$ and $s, s_1, s_2 \in L_2$, the Cb-CCLM has the following properties:*

(1) If $p_1 \leqslant p_2$, then $F(p_1) \leqslant F(p_2)$.

(2) If $s_1 \leqslant s_2$, then $G(s_1) \leqslant G(s_2)$.

(3) $F(p_1 \wedge p_2) \leqslant F(p_1) \wedge F(p_2)$.

(4) $G(s_1 \vee s_2) \geqslant G(s_1) \vee G(s_2)$.

(5) $F \circ G \circ F(p) = F(p)$.

(6) $G \circ F \circ G(s) = G(s)$.

(7) $F(p) \leqslant s \Leftrightarrow p \leqslant G(s)$.

Proof. (1) For any $p_1, p_2 \in L_1$, if $p_1 \leqslant p_2$, by Definition 3.1, then $F(p_2) = F(p_1 \vee p_2) = F(p_1) \vee F(p_2)$, Therefore, $F(p_1) \leqslant F(p_2)$.

(2) It can be proven similarly to (1).

(3) For $p_1 \wedge p_2 \leqslant p_1$ and $p_1 \wedge p_2 \leqslant p_2$, by (1), then $F(p_1 \wedge p_2) \leqslant F(p_1)$, and $F(p_1 \wedge p_2) \leqslant F(p_2)$. Thus, $F(p_1 \wedge p_2) \leqslant F(p_1) \wedge F(p_2)$.

(4) It can be proven similarly to (3).

(5) For (L_1, L_2, F, G) is a Cb-CCLM, by Definition 3.2, there is $G \circ F(p) \geqslant p$, then $F \circ G \circ F(p) \geqslant F(p)$ by (1). For $F \circ G(s) \leqslant s$, we take $s = F(p)$, then $F \circ G \circ F(p) \leqslant F(p)$. Thus, $F \circ G \circ F(p) = F(p)$.

(6) We can proof it similarly to (5).

(7) If $F(p) \leqslant s$, then $G \circ F(p) \leqslant G(s)$ by (2). By Definition 3.2, then $G \circ F(p) \geqslant p$. Therefore, $p \leqslant G(s)$. In the similar manner, if $p \leqslant G(s)$, then $F(p) \leqslant s$. Thus, $F(p) \leqslant s \Leftrightarrow p \leqslant G(s)$.

Proposition 3.2. *Let (I, T, R) be a skill context, where $I = \{i_1, i_2, \ldots, i_n\}$ and $T = \{t_1, t_2, \ldots, t_m\}$. If $L_1 = 2^I$ and $L_2 = 2^T$, then the operators "$\diamond$" and "$\Box$" defined by Definition 2.2 are the two dual competence-based cognitive learning operators of (I, T, R).*

Proof. For $p_1, p_2 \in L_1$, and $s_1, s_2 \in L_2$, we can obtain that the operators "$\diamond$" and "$\Box$" satisfy the properties:

(1) $\emptyset^\diamond = \emptyset, I^\diamond = T$, (2) $(p_1 \cup p_2)^\diamond = p_1^\diamond \cup p_2^\diamond$, (3) $\emptyset^\Box = \emptyset, T^\Box = I$, (4) $(s_1 \cap s_2)^\Box = s_1^\Box \cap s_2^\Box$.

Then by Definition 3.1, the operators "$\diamond$" and "$\Box$" are the two dual competence-based cognitive learning operators of (I, T, R).

Remark 3.2. From Proposition 3.2, "$\diamond$" and "$\Box$" are the two operators of Definition 3.1 in a skill context. With the results of Lemma 2.4 and Proposition 2.1, if (P, S) is a Cb-concept, then P is the maximal item set induced by the minimal skill set S. And the skills of S are called necessary and sufficient for P from the perspective of Cb-concepts.

Example 3.1. Let $I = \{i_1, i_2, i_3, i_4\}$, $T = \{r, t, v, s\}$, where $i_1 : 3+4$, $i_2 : 3 \times 2 - 5$, $i_3 : \frac{10}{2} + 3$, $i_4 : \frac{1}{10} + \frac{2}{5}$, r :addition, t : subtraction, v : multiplication, and s : division. The conjunctive model of skill map (I, T, τ) is to calculate the value of the items, where $\tau(i_1) = \{r\}$, $\tau(i_2) = \{t, v\}$, $\tau(i_3) = \{r, s\}$, $\tau(i_4) = \{r, v, s\}$. The competence-based cognitive learning operators F and G are taken as "$\diamond$" and "$\Box$" in this Cb-CCLM. For $\{r, s\}^\Box = \{i_1, i_3\}$ and $\{i_1, i_3\}^\diamond = \{r, s\}$, then $(i_1 i_3, rs)$ is a Cb-concept. Therefore, the skills r and s are the least amount skills for solving the item set $\{i_1, i_3\}$, and $\{i_1, i_3\}$ has the maximal amount items solved by the skill set $\{r, s\}$. And the skill set $\{r, s\}$ is sufficient and necessary for solving $\{i_1, i_3\}$. Otherwise, for the skill set $\{r\}$ is needed, but not enough for solving $\{i_1, i_3\}$. Then, $\{r\}$ is a sufficient skill set for $\{i_1, i_3\}$. For the skill set $\{r, t, s\}$ is too much for solving $\{i_1, i_3\}$, where the skill t is not needed, then $\{r, t, s\}$ is called as a necessary skill set of $\{i_1, i_3\}$. The skill set $\{v\}$ can not solve $\{i_1, i_3\}$, then $\{v\}$ is nether sufficient nor necessary for solving $\{i_1, i_3\}$. Thus, if the item set P and the skill set S have not been a Cb-concept, then the skills of S may be too much, too little, or unable to solve the item set P. Thus, the skills of S are necessary, sufficient, or neither necessary nor sufficient for P. At this time, one does not complete the cognitive learning. In conclusion, the operators "$\diamond$" and "$\Box$" can reflect the relationship between knowledge and skills in a skill context, and there is a necessary and sufficient relationship between knowledge and skills.

4. Transformation between competence-based information granules in a Cb-CCLM

4.1. Competence-based information granules of a Cb-CCLM

As analyzed above, there is a necessary and sufficient relationship between knowledge and skills. If the skills are not necessary and sufficient for the problems, then one does not obtain the minimal skill set for the most knowledge. In the concept-cognitive learning processes, the knowledge and skills will change. In essence, the cognitive learning processes of Cb-CCLM can be reflected by the changes between the item sets and the skill sets. In this section, we will discuss the relationship between knowledge and skills in the cognitive process with granular computing. Then, an information granule structure is constructed by the sufficient and necessary relationship between skills and knowledge in the Cb-CCLM. To characterize the granule description of the Cb-CCLM, a pair (p, s) is denoted as an information granule, where p is an item set and s is a skill set.

Definition 4.1. Let (L_1, L_2, F, G) be a Cb-CCLM. For any $p \in L_1$ and $s \in L_2$, we denote two sets as follows:

$\mathcal{N} = \{(p, s) | s \geqslant F(p), p \leqslant G(s)\}$,

$\mathcal{S} = \{(p, s) | s \leqslant F(p), p \geqslant G(s)\}$.

(1) If $(p, s) \in \mathcal{N}$, then (p, s) is a necessary competence-based information granule of this Cb-CCLM, s is a necessary skill set of p, and $\mathcal{N}$ is a necessary competence-based information granules set of this Cb-CCLM.

(2) If $(p, s) \in \mathcal{S}$, then (p, s) is a sufficient competence-based information granule of this Cb-CCLM, s is a sufficient skill set of p, and $\mathcal{S}$ is a sufficient competence-based information granules set of this Cb-CCLM.

(3) If $(p, s) \in \mathcal{N} \cap \mathcal{S}$, i.e., (p, s) satisfies $s = F(p)$ and $p = G(s)$, then (p, s) is a sufficient and necessary competence-based information granule of this Cb-CCLM, and s is a sufficient and necessary skill set of p. In this case, (p, s) is a competence-based cognitive concept.

(4) If $(p, s) \in \mathcal{N} \cup \mathcal{S}$, then (p, s) is a consistent information granule of this Cb-CCLM.

(5) If $(p, s) \notin \mathcal{N} \cap \mathcal{S}$, then (p, s) is an inconsistent information granule of this Cb-CCLM.

Remark 4.1. (1) If an information granule (p, s) satisfies Proposition 3.1 (7), then $(p, s) \in \mathcal{N}$. If an information granule (p, s) satisfies the inverse negative statement of Proposition 3.1 (7), then $(p, s) \in \mathcal{S}$. Therefore, only information granule sets $\mathcal{N}$ and $\mathcal{S}$ are used to categorize the information granules. (2) If an information granule $(p, s) \in \mathcal{N}$, then $p \leqslant G(s)$ and $s \geqslant F(p)$. Thus, the items solved by s will be more than the items of p, and the more effective skills [50] will be required for solving them. Only when one learns and masters the effective skills, the item set will change. It means that s is a necessary skill set for solving p. If an information granule $(p, s) \in \mathcal{S}$, then $s \leqslant F(p)$ and $p \geqslant G(s)$. Similarly, s is a sufficient skill set for solving p. If an information granule (p, s) is a necessary competence-based information granule or sufficient competence-based information granule, then (p, s) is a consistent information granule. If an information granule $(p, s) \notin \mathcal{N} \cap \mathcal{S}$, then the skill set s is neither sufficient nor necessary for solving p, and (p, s) is an inconsistent information granule.

Example 4.1. For the conjunctive model of skill map (I, T, τ) of Example 3.1, the information granule (i_1, rv) means that the item set is $\{i_1\}$ and the skill set is $\{r, v\}$. In fact, if the skill set is $\{r, v\}$, then one can only solve $\{i_1\}$. But if the item set is $\{i_1\}$, the minimal skill set is $\{r\}$. Therefore, (i_1, rv) is not a Cb-concept. For $\{i_1\}^\diamond = \{r\} \subseteq \{r, v\}$ and $\{r, v\}^\Box = \{i_1\} \subseteq \{i_1\}$. Then, the information granule (i_1, rv) is a necessary competence-based information granule, and $\{r, v\}$ is a necessary skill set for solving $\{i_1\}$, where the skill v is not needed to solve $\{i_1\}$. For the information granule $(i_1 i_2, r)$, where $\{i_1, i_2\}^\diamond = \{r, t, v\} \supseteq \{r\}$ and $r^\Box = \{i_1\} \subseteq \{i_1, i_2\}$, then $(i_1 i_2, r)$ is a sufficient competence-based information granule, and $\{r\}$ is a sufficient skill set for $\{i_1, i_2\}$. Thus, (i_1, rv) and $(i_1 i_2, r)$ are consistent competence-based information granules. For the information granule (i_3, rv), where $\{i_3\}^\diamond = \{r, s\} \supsetneq \{r, v\}$ and $\{i_3\}^\diamond \subsetneq \{r, v\}$, then (i_3, rv) is neither a sufficient competence-based information granule, nor a necessary competence-based information granule, and $\{r, v\}$ is neither sufficient nor necessary for $\{i_3\}$. Thus, (i_3, rv) is an inconsistent competence-based information granule. In conclusion, the information granules $(i_1, rv), (i_1 i_2, r), (i_3, rv)$ are not sufficient and necessary competence-based information granules, i.e., they are not competence-based cognitive concepts.

In general, sufficient and necessary competence-based information granules do not exist at the beginning of cognitive learning process. One may start from an unknown knowledge or skills, or an inconsistent information granule, then one can try to obtain consistent information granules, which may be necessary or sufficient competence-based information granules. Through further cognitive learning with skills, one can acquire some sufficient and necessary competence-based information granules.

Through Proposition 3.2, a sufficient and necessary competence-based information granule of Cb-CCLM is actually a Cb-concept in a skill context.

Proposition 4.1. *Let (I, T, R) be a skill context, where $I = \{i_1, i_2, \ldots, i_n\}$ and $T = \{t_1, t_2, \ldots, t_m\}$. For any $P \subseteq I$ and $S \subseteq T$, then*

(1) If $P^{\diamond} \subseteq S$ and $P \subseteq S^{\Box}$, then S is a necessary skill set of P.

(2) If $P^{\diamond} \supseteq S$ and $P \supseteq S^{\Box}$, then S is a sufficient skill set of P.

(3) If $P^{\diamond} = S$ and $P = S^{\Box}$, then S is a sufficient and necessary skill set of P.

Proof. It is obtained by Definition 4.1 and Proposition 3.2.

Proposition 4.2. *Let (I, T, R) be a skill context. For $P \subseteq I$ and $S \subseteq T$,*

(1) If $P^{\diamond} \subseteq S$, $P \subseteq S^{\Box}$, then $(P \bigcup S^{\Box}, (P \bigcup S^{\Box})^{\diamond})$, $((S \bigcap P^{\diamond})^{\Box}, S \bigcap P^{\diamond})$ are Cb-Concepts.

(2) If $P^{\diamond} \supseteq S$, $P \supseteq S^{\Box}$, then $(P \bigcap S^{\Box}, (P \bigcap S^{\Box})^{\diamond})$, $((S \bigcup P^{\diamond})^{\Box}, S \bigcup P^{\diamond})$ are Cb-Concepts.

Proof. We can proof it through Definition 2.3.

For $(p_1, s_1), (p_2, s_2) \in \mathcal{N} \bigcup \mathcal{S}$, "$\preceq$" is defined with $(p_1, s_1) \preceq (p_2, s_2) \Leftrightarrow p_1 \leqslant p_2$ $(s_1 \leqslant s_2)$. The operators "$\wedge_{\mathcal{G}}$" and "$\vee_{\mathcal{G}}$" are defined, and

$(p_1, s_1) \wedge_{\mathcal{G}} (p_2, s_2) = (p_1 \wedge p_2, F \circ G(s_1 \wedge s_2))$,

$(p_1, s_1) \vee_{\mathcal{G}} (p_2, s_2) = (G \circ F(p_1 \vee p_2), s_1 \vee s_2)$.

Proposition 4.3. *Let (L_1, L_2, F, G) be a Cb-CCLM, $\mathcal{N}$, $\mathcal{S}$ be a necessary and a sufficient competence-based information granules set of this Cb-CCLM, respectively. Then,*

(1) $(\mathcal{N}, \preceq)$ is closed under the operators "$\vee_{\mathcal{G}}$" and "$\wedge_{\mathcal{G}}$".

(2) $(\mathcal{S}, \preceq)$ is closed under the operators "$\vee_{\mathcal{G}}$" and "$\wedge_{\mathcal{G}}$".

Proof. (1) Let $(p_1, s_1), (p_2, s_2) \in \mathcal{N}$, then $s_1 \geqslant F(p_1)$, $p_1 \leqslant G(s_1)$, $s_2 \geqslant F(p_2)$ and $p_2 \leqslant G(s_2)$. Then $p_1 \wedge p_2 \leqslant G(s_1) \wedge G(s_2) = G(s_1 \wedge s_2) = G \circ F \circ G(s_1 \wedge s_2)$. Moreover, using Definition 3.1 and Proposition 3.1, there is $F(G(s_1 \wedge s_2)) = F(G(s_1) \wedge G(s_2)) \geqslant F(p_1 \wedge p_2)$, Thus, $(p_1, s_1) \wedge_{\mathcal{G}} (p_2, s_2) \in \mathcal{N}$. Then, $(p_1, s_1) \vee_{\mathcal{G}} (p_2, s_2) \in \mathcal{N}$ can be proven similarly.

(2) Let $(p_1, s_1), (p_2, s_2) \in \mathcal{S}$, then $s_1 \leqslant F(p_1)$, $p_1 \geqslant G(s_1)$, $s_2 \leqslant F(p_2)$ and $p_2 \geqslant G(s_2)$, then $p_1 \wedge p_2 \geqslant G(s_1) \wedge G(s_2) = G(s_1 \wedge s_2) = G \circ F \circ G(s_1 \wedge s_2)$. Moreover, using Definition 3.1 and Proposition 3.1, then $F(G(s_1 \wedge s_2)) = F(G(s_1) \wedge G(s_2)) \leqslant F(p_1 \wedge p_2)$. Thus, $(p_1, s_1) \wedge_{\mathcal{G}} (p_2, s_2) \in \mathcal{S}$. Then, $(p_1, s_1) \vee_{\mathcal{G}} (p_2, s_2) \in \mathcal{S}$ can be proven similarly.

According to Proposition 4.3, "$\preceq$" is a quasi-order relationship in $(\mathcal{N}, \preceq)$ and $(\mathcal{S}, \preceq)$ with respect to the operators "$\wedge_{\mathcal{G}}$" and "$\vee_{\mathcal{G}}$". Therefore, $(\mathcal{N}, \preceq)$ and $(\mathcal{S}, \preceq)$ are "quasi lattices". In Example 2.2, part of necessary competence-based information granules of $\mathcal{N}$ are shown in Fig. 2, part of sufficient competence-based information granules of $\mathcal{S}$ are shown in Fig. 3.

4.2. Transformation of information granules based on a conjunctive model of skill map

In the Cb-CCLM, one always begins to learn from unknown knowledge or skills, or one does not have consistent information granules at the beginning of CCLM. In general, one can start from indirect relationship between the item sets and the skill

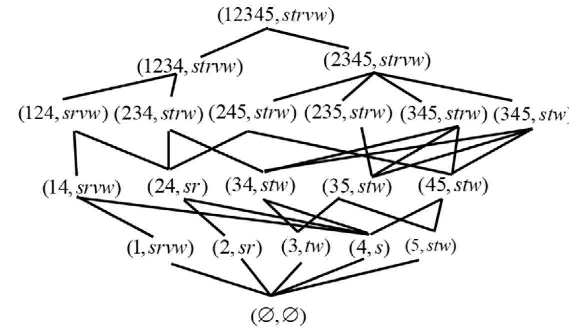

Fig. 2. Part of necessary competence-based information granules in $\mathcal{N}$.

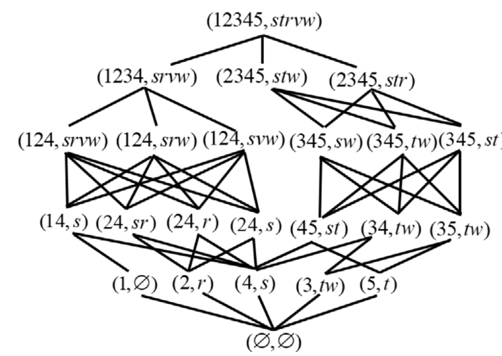

Fig. 3. Part of sufficient competence-based information granules in $\mathcal{S}$.

sets, and try to acquire some necessary or sufficient competence-based information granules in the concept-cognitive learning process, then finally obtain the sufficient and necessary competence-based information granules. Therefore, a transformation method between information granules is presented to convert a general information granule into sufficiency and necessary competence-based information granules.

Proposition 4.4. *Let (L_1, L_2, F, G) be a Cb-CCLM, and $\mathcal{N}$ be a necessary competence-based information granules set of this Cb-CCLM. If $p \in L_1$ and $s \in L_2$, then*

(N1) $(p \wedge G(s), F(p) \wedge s) \in \mathcal{N}$.

(N2) $(p \vee G(s), F(p) \vee s) \in \mathcal{N}$.

(N3) $(p \wedge G(s), F(p)) \in \mathcal{N}$.

(N4) $(G(s), F(p) \vee s) \in \mathcal{N}$.

(N5) $(G \circ F(p), F(p) \vee s) \in \mathcal{N}$.

(N6) $(p \wedge G(s), F \circ G(s)) \in \mathcal{N}$.

Proof. (N1) Because (L_1, L_2, F, G) is a Cb-CCLM, from Definitions 3.1, 3.2, 4.1, and Proposition 3.1, there are: $F(p \wedge G(s)) \leqslant F(p) \wedge F(G(s)) \leqslant F(p) \wedge s$, and $G(F(p) \wedge s) = G(F(p)) \wedge G(s) \geqslant p \wedge G(s)$. Then $(p \wedge G(s), F(p) \wedge s) \in \mathcal{N}$.

(N2) We can proof it similarly to (N1).

(N3) Because (L_1, L_2, F, G) is a Cb-CCLM, from Definitions 3.1, 3.2, 4.1, and Proposition 3.1, there are: $F(p \wedge G(s)) \leqslant F(p) \wedge F(G(s)) \leqslant F(p)$, and $G(F(p)) \geqslant p \geqslant p \wedge G(s)$. Then $(p \wedge G(s), F(p)) \in \mathcal{N}$.

(N4) We can proof it similarly to (N3).

(N5) Because (L_1, L_2, F, G) is a Cb-CCLM, from Definitions 3.1, 3.2, 4.1, and Proposition 3.1, there are: $F \circ G \circ F(p) = F(p) \leqslant F(p) \vee s$, and $G(F(p) \vee s) \geqslant G(F(p)) \vee G(s) \geqslant G \circ F(p)$. Then $(G \circ F(p), F(p) \vee s) \in \mathcal{N}$.

(N6) We can proof it similarly to (N5).

Proposition 4.5. *Let (L_1, L_2, F, G) be a Cb-CCLM, and $\mathcal{S}$ be a sufficient competence-based information granules set of this Cb-CCLM. If $p \in L_1$ and $s \in L_2$, then*
(S1) $(G \circ F(p), F(p) \wedge s) \in \mathcal{S}$.
(S2) $(p \vee G(s), F \circ G(s)) \in \mathcal{S}$.

Proof. (S1) Because (L_1, L_2, F, G) is a Cb-CCLM, from Definitions 3.1, 3.2 and 4.1, and Proposition 3.1, there are: $F \circ G \circ F(p) = F(p) \geqslant F(p) \wedge s$, and $G(F(p) \wedge s) = G(F(p)) \wedge G(s) \leqslant G \circ F(p)$. Then $(G \circ F(p), F(p) \wedge s) \in \mathcal{S}$.
(S2) We can proof it similarly to (S1).

Proposition 4.6. *Let (L_1, L_2, F, G) be a Cb-CCLM, $\mathcal{N}$, $\mathcal{S}$ be a necessary and a sufficient competence-based information granules set of this Cb-CCLM, respectively. If $p \in L_1$, $s \in L_2$ and $(p, s) \in \mathcal{N}$, then*
(NS1) $(G(F(p) \wedge s), F(p) \wedge s) \in \mathcal{N} \cap \mathcal{S}$.
(NS2) $(p \vee G(s), F(p \vee G(s))) \in \mathcal{N} \cap \mathcal{S}$.

Proof. (NS1) If $(p, s) \in \mathcal{N}$, then $F(p) \leqslant s$ and $G(s) \geqslant p$. Thus, $F(p) \wedge s = F(p)$, and $G(F(p) \wedge s) = G \circ F(p)$. Because (L_1, L_2, F, G) is a Cb-CCLM, from Definition 4.1 and Proposition 3.1, there is: $F(G(F(p) \wedge s)) = F \circ G \circ F(p) = F(p) = F(p) \wedge s$. Thus, $(G(F(p) \wedge s), F(p) \wedge s) \in \mathcal{S}$. Then, $(G(F(p) \wedge s), F(p) \wedge s) \in \mathcal{N} \cap \mathcal{S}$.
(NS2) We can proof it similarly to (NS1).

Proposition 4.7. *Let (L_1, L_2, F, G) be a Cb-CCLM, $\mathcal{N}$, $\mathcal{S}$ be a necessary and a sufficient competence-based information granules set of this Cb-CCLM, respectively. If $p \in L_1$, $s \in L_2$ and $(p, s) \in \mathcal{S}$, then*
(SN1) $(G(F(p) \vee s), F(p) \vee s) \in \mathcal{N} \cap \mathcal{S}$.
(SN2) $(p \wedge G(s), F(p \wedge G(s))) \in \mathcal{N} \cap \mathcal{S}$.

Proof. (SN1) If $(p, s) \in \mathcal{S}$, then $F(p) \geqslant s$ and $G(s) \leqslant p$. Thus, $F(p) \vee s = F(p)$ and $G(F(p) \vee s) = G(F(p))$. Because (L_1, L_2, F, G) is a Cb-CCLM, from Definition 4.1 and Proposition 3.1, there is: $F(G(F(p) \vee s)) = F \circ G \circ F(p) = F(p) = F(p) \vee s$. Thus, $(G(F(p) \vee s), F(p) \vee s) \in \mathcal{N}$. Then, $(G(F(p) \vee s), F(p) \vee s) \in \mathcal{N} \cap \mathcal{S}$.
(SN2) We can proof it similarly to (SN1).

It is easy to find that there are two paths which can convert an inconsistent information granule into sufficient and necessary competence-based information granules. A path is called concept-cognitive learning path 1 (CCLP-1), if necessary competence-based information granules can be firstly gained from an inconsistent competence-based information granule through six approaches (N1)∼(N6), and they can be ultimately transformed into sufficient and necessary competence-based information granules by two methods (NS1)∼(NS2). Concept-cognitive learning path 2 (CCLP-2) is called that if an inconsistent information granule can be converted into sufficient competence-based information granules by (S1)∼(S2), and they can be furthermore converted into sufficient and necessary competence-based information granules by (SN1)∼(SN2). The processes of CCLP-1 and CCLP-2 are shown in Fig. 4 and 5, respectively.

In the CCLP-1, the number of sufficient and necessary competence-based information granules generated from an inconsistent granule is less than 12, while some of them are the same. In the same way, it is easy to get that there are less than 4 Cb-concepts generated through CCLP-2. Thus, the total number of sufficient and necessary competence-based information granules is less than 16.

Remark 4.2. Essentially, the pair (F, G) is $(\diamond, \Box)$ in a skill context (I, T, R). Then, F and G are taken as "$\diamond$" and "$\Box$" in the following algorithms, respectively. In the skill context (I, T, R), sufficient and necessary competence-based information granules are Cb-concepts. Then, the concept-cognitive learning process of a Cb-CCLM based on the conjunctive model of skill map is shown in Algorithm 1. The flow diagram of Algorithm 1 is shown in Fig. 6.

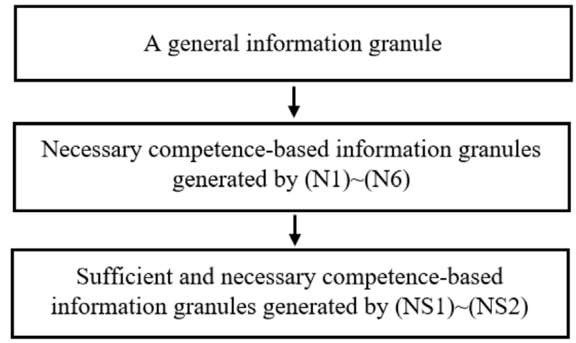

Fig. 4. Transformation between information granules by CCLP-1.

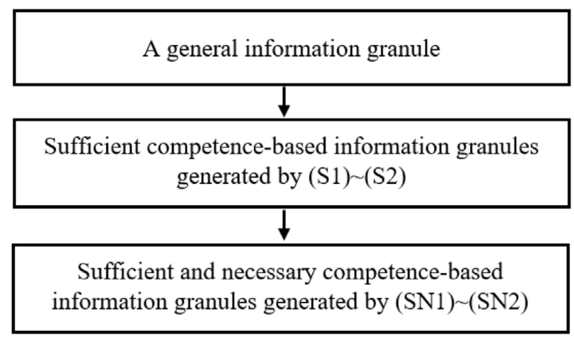

Fig. 5. Transformation between information granules by CCLP-2.

Table 2
The skill mastery of knowledge state in $(I_2, \mathcal{P}_2)$.

$\mathcal{P}_2 \backslash T_2$	t	s	r	v	w
$\emptyset$	×	×			×
{3}	✓	×			✓
{4}	×	✓	×		×
{2, 4}	×	✓	✓	×	×
{3, 4, 5}	✓	✓	×		✓
{1, 2, 4}	×	✓	✓	✓	✓
{2, 3, 4, 5}	✓	✓	✓	×	✓
{1, 2, 3, 4, 5}	✓	✓	✓	✓	✓

Remark 4.3. The time complexity of Algorithm 1 is analyzed step by step. The numbers of items and skills are denoted by $|I|$ and $|T|$, respectively. Running step 3 takes $O(max(|I||T| + |I|, |I||T| + |T|))$, running step 5 takes $O(|I||T| + |I| + |T|)$, step 6 takes $O(max(|I||T| + |I|, |I||T| + |T|))$, so is that of steps 8–9, step 11 and step 13. Thus, the computational complexity of Algorithm 1 is $O(|I||T| + |I| + |T|)$.

Example 4.2. Let $p_0 = \{3\}, s_0 = \{t, s\}$ in Example 2.2. For the information granule $(3, ts)$, where $\{3\}^{\diamond} = \{t, w\} \supsetneq \{t, s\}$, $\{3\}^{\diamond} \subsetneq \{t, s\}$, $\{t, s\}^{\Box} = \{4\} \supsetneq \{3\}$ and $\{t, s\}^{\Box} \subsetneq \{3\}$, then $(3, ts)$ is an inconsistent information granule. By CCLP-1, we can transform $(3, ts)$ into the necessary competence-based information granules $(\emptyset, t), (\emptyset, tw), (\emptyset, s), (3, stw), (4, stw), (34, stw)$ by (N1)∼(N6), then convert them into sufficient and necessary competence-based information granules $(\emptyset, \emptyset), (4, s), (3, tw), (345, stw)$ by (NS1)∼(NS2). In the other way, by CCLP-2, $(3, ts)$ is transformed into the sufficient competence-based information granules $(3, t), (34, s)$ by (S1)∼(S2), and they can be further transformed into sufficient and necessary competence-based information granules $(\emptyset, \emptyset), (4, s), (3, tw), (345, stw)$ by (SN1)∼(SN2).

Algorithm 1 : *Transformation of information granules in a Cb − CCLM based on the conjunctive model of skill map*

Input : The conjunctive model of skill map (I, T, τ) and a general information granule (p_0, s_0) in the Cb-CCLM.
Output : Sufficient and necessary competence-based information granules.

$(I, T, R) \leftarrow (I, T, \tau)$.
//Transform (p_0, s_0) into sufficient and necessary competence-based information granules.
if $(p_0, s_0) \notin \mathcal{N} \cup \mathcal{S}$
  // Through CCLP-1
  $(p_0, s_0) \leftarrow (p_1^1, s_1^1), \dots, (p_1^m, s_1^m)$, $m \leqslant 6$ by (N1)∼(N6).
  $(p_1^1, s_1^1), \dots, (p_1^m, s_1^m) \leftarrow (p_2^1, s_2^1), \dots, (p_2^n, s_2^n)$, $n \leqslant 12$ by (NS1)∼(NS2).
  // Through CCLP-2
  $(p_0, s_0) \leftarrow (p_3^1, s_3^1), \dots, (p_3^h, s_3^h)$, $h \leqslant 2$ by (S1)∼(S2).
  $(p_3^1, s_3^1), \dots, (p_3^h, s_3^h) \leftarrow (p_4^1, s_4^1), \dots, (p_4^t, s_4^t)$, $t \leqslant 4$ by (SN1) ∼(SN2).
else if $(p_0, s_0) \in \mathcal{N}$
    $(p_0, s_0) \leftarrow (p_5^1, s_5^1), (p_5^2, s_5^2)$ by (NS1)∼(NS2).
  else
    $(p_0, s_0) \leftarrow (p_6^1, s_6^1), (p_6^2, s_6^2)$ by (SN1)∼(SN2).
  end if
end if
// Return different sufficient and necessary competence-based information granules
$(p^1, s^1), \dots, (p^l, s^l) \leftarrow (p_2^1, s_2^1), \dots, (p_2^n, s_2^n), (p_4^1, s_4^1), \dots, (p_4^t, s_4^t)$,
or $(p_5^1, s_5^1), (p_5^2, s_5^2)$, or $(p_6^1, s_6^1), (p_6^2, s_6^2)$.

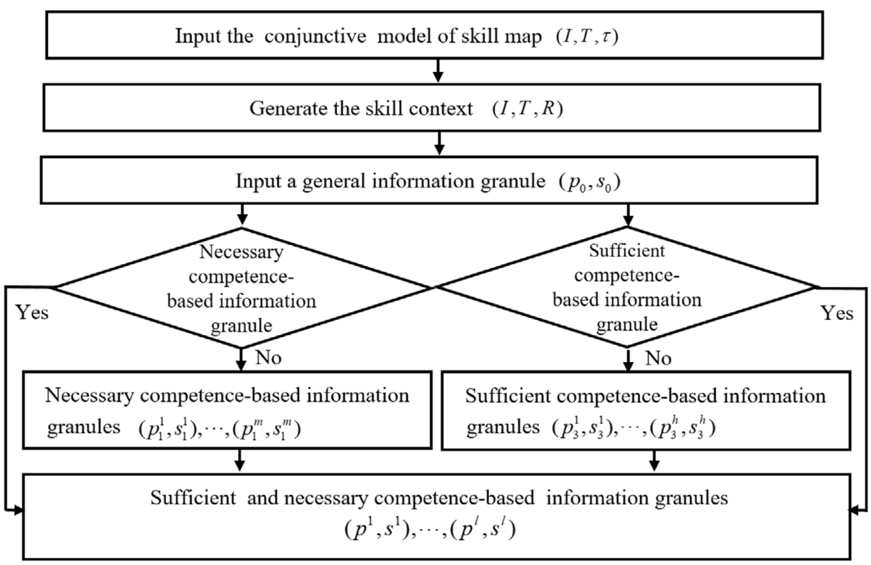

Fig. 6. The flow diagram of Algorithm 1.

Thus, through the two concept-cognitive learning paths, an inconsistent information granule can be converted into necessary, sufficient, necessary and sufficient competence-based information granules (i.e., Cb-concepts).

From the skill context (I_2, T_2, R), we can obtain the mastery of skills according to [48], which is shown in Table 2. In Table 2, "$P \checkmark s$" means that one with the knowledge state P has mastered the skill s, and "$P \times s$" means that one with the knowledge state P does not master the skill s, where $P \in \mathcal{P}_2$ and $s \in T_2$.

Thus, if one owns the information granule $(3, st)$, then one does not master the skill s. Otherwise, one can solve $\{s\}^\square = \{4\}$. Furthermore, if one does not master the skill t and $\{3\}^\diamond = \{t, w\}$, then one can not solve $\{3\}$. Therefore, one with the information granule $(3, st)$ has mastered the skill set $\{t\}$, but not the skill set $\{s\}$. In the cognitive learning process, the existing skills can be consolidated to make the basic foundation more solid, and new skills can be learned to solve new items. For the information granule $(3, st)$ can be transformed into four Cb-concepts $(\emptyset, \emptyset), (4, s), (3, tw), (345, stw)$ in the concept-cognitive learning. Therefore, one can learn and master the skill s to get the Cb-concept $(4, s)$. Or one can consolidate the skill t, learn and master the skill w to get the Cb-concept $(3, tw)$. Or one can consolidate the skill t, learn and master the skills s, w to obtain the Cb-concept $(345, stw)$. In this case, one has three learning paths for skill.

Similarly, another information granule $(24, svw)$ can be transformed into sufficient and necessary competence-based information granules $(\emptyset, \emptyset)$, $(4, s)$, $(24, sr)$, $(124, srvw)$ by CCLP-1. Through CCLP-2, $(24, svw)$ can be transformed into sufficient and necessary competence-based information granules $(\emptyset, \emptyset)$, $(4, s)$, $(24, sr)$. For the information granule $(24, svw)$, and $\{2\}^\diamond \cap \{4\}^\diamond = \{s\}$, if one does not mastered the skill s, then one can not solve $\{2\}$ or $\{4\}$. Thus, one has master at least the skill s. However, for $\{2, 4\}^\diamond = \{s, r\}$ and the skills v, w are not needed to solve $\{2, 4\}$, then we can not deduce whether one has learned and mastered the skills v, w. Thus, one with the granule $(24, svw)$ has mastered the skill set $\{s\}$ and solved $\{4\}$. And then, one can consolidate the skill s, learn and master the skill r to obtain the Cb-concept

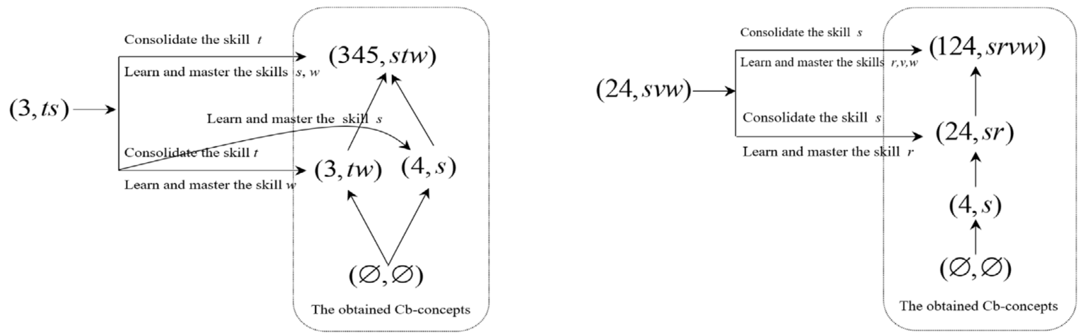

Fig. 7. The skill learning paths diagram obtained from information granules.

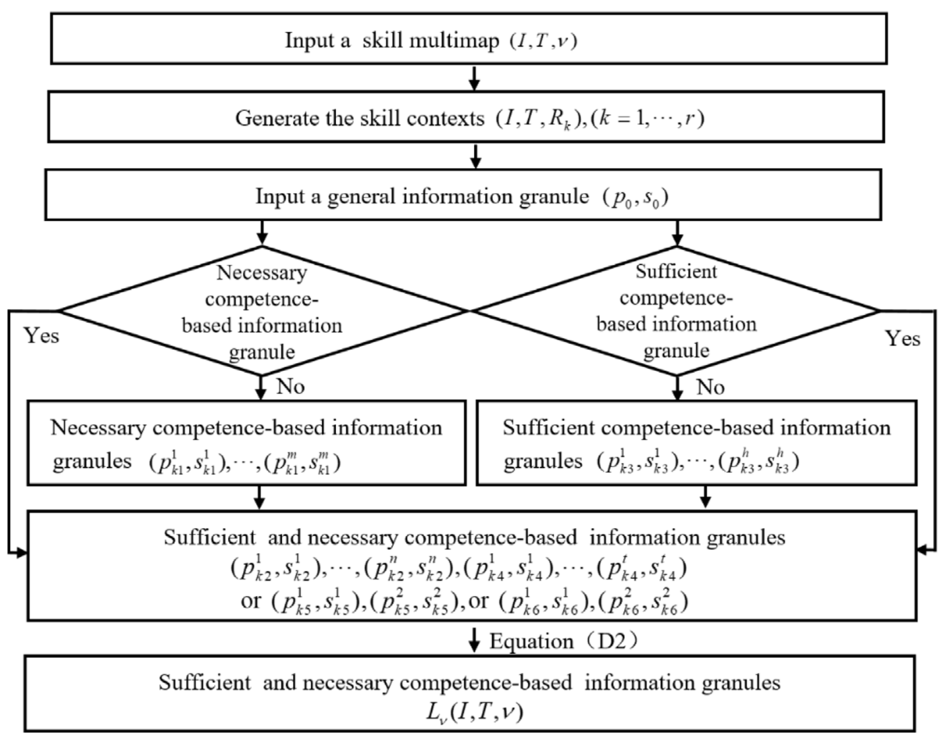

Fig. 8. The flow diagram of Algorithm 2.

$(24, sr)$. Or one can consolidate the skill s, learn and master the skills r, v, w to acquire the Cb-concept $(124, srvw)$. Thus, one with the information granule $(24, svw)$ has two learning paths for skill. The skill learning paths of information granules $(3, ts)$ and $(24, svw)$ are shown in Fig. 7, respectively.

According to Ausubel, concept learning is an advanced form of meaningful learning, and it is to master the common key characteristics of similar things essentially. Any meaningful learning involves the transfer of the cognitive structure [51]. Through the transformation between information granules of the Cb-CCLM, one can obtain several sufficient and necessary competence-based information granules (i.e., Cb-concepts) transformed from a general information granule, and the obtained Cb-concepts can guide one to learn further. In fact, (1) if the initial information granule is a Cb-concept, then the number of Cb-concepts obtained by Algorithm 1 is only one, which is the initial information granule itself. At this time, one has obtained the sufficient and necessary competence-based information granule, and one has grasped the sufficient and necessary skills for the knowledge. (2) If the initial information granule is a sufficient or necessary competence-based information granule, then the number of the obtained Cb-concepts is two. Therefore, one has two learning schemes to learn further. (3) If the initial information granule is an inconsistent information granule, then the number of the obtained Cb-concepts is less than 16. Thus, one has more choices for learning further. One can choose a desired learning path for skill, and then one should learn, do some practices or tests with the skills, and master them. In general, individuals with different initial information granules will obtain different competence-based cognitive concepts and learning paths for skill, such as the two information granules $(3, ts)$ and $(24, svw)$. Thus, the obtained Cb-concepts can guide one to realize personalized learning. If the number of the obtained Cb-concepts is less, it is easier to choose and obtain the learning paths for skill.

In conclusion, Algorithm 1 proposed in the paper can obtain the cognitive concepts from the local subset of the concept lattice, rather than the whole concept lattice or all granular concepts. And we mainly focus on Algorithm 1 to obtain the Cb-concepts

Algorithm 2 : *Transformation of information granules in a Cb − CCLM based on a skill multimap*

Input : A skill multimap (I, T, υ) and a general information granule (p_0, s_0) in the Cb-CCLM.

Output : Sufficient and necessary competence-based information granules.

(I, T, R_k) $(k = 1, 2, \cdots, r) \leftarrow (I, T, \upsilon)$, where $r = \prod_{p \in I} |\upsilon(p)|$.
//Transform (p_0, s_0) into sufficient and necessary competence-based information granules in (I, T, R_k).
for $k = 1 : r$
 if $(p_0, s_0) \notin \mathcal{N}_k \cup \mathcal{S}_k$
  // Through CCLP-1
  $(p_0, s_0) \leftarrow (p_{k1}^1, s_{k1}^1), \ldots, (p_{k1}^m, s_{k1}^m)$, $m \leqslant 6$ by (N1)∼(N6).
  $(p_{k1}^1, s_{k1}^1), \ldots, (p_{k1}^m, s_{k1}^m) \leftarrow (p_{k2}^1, s_{k2}^1), \ldots, (p_{k2}^n, s_{k2}^n)$, $n \leqslant 12$ by (NS1)∼(NS2).
  // Through CCLP-2
  $(p_0, s_0) \leftarrow (p_{k3}^1, s_{k3}^1), \ldots, (p_{k3}^h, s_{k3}^h)$, $h \leqslant 2$ by (S1)∼(S2).
  $(p_{k3}^1, s_{k3}^1), \ldots, (p_{k3}^h, s_{k3}^h) \leftarrow (p_{k4}^1, s_{k4}^1), \ldots, (p_{k4}^t, s_{k4}^t)$, $t \leqslant 4$ by (SN1) ∼(SN2).
 else if $(p_0, s_0) \in \mathcal{N}_k$
   $(p_0, s_0) \leftarrow (p_{k5}^1, s_{k5}^1), (p_{k5}^2, s_{k5}^2)$ by (NS1)∼(NS2).
  else
   $(p_0, s_0) \leftarrow (p_{k6}^1, s_{k6}^1), (p_{k6}^2, s_{k6}^2)$ by (SN1)∼(SN2).
  end if
 end if
end for
$L_\upsilon(I, T, \upsilon) \leftarrow (p_{k2}^1, s_{k2}^1), \ldots, (p_{k2}^n, s_{k2}^n), (p_{k4}^1, s_{k4}^1), \ldots, (p_{k4}^t, s_{k4}^t)$,
or $(p_{k5}^1, s_{k5}^1), (p_{k5}^2, s_{k5}^2)$,or $(p_{k6}^1, s_{k6}^1), (p_{k6}^2, s_{k6}^2)$, $(k = 1, 2, \cdots, r)$ through (D2).
Return $L_\upsilon(I, T, \upsilon)$

Table 3
UCI data sets.

No	Data sets	$\|I\|$	$\|T\|$
1	Zoo	101	17
2	Indian Liver Patient	583	9
3	Vowel-Context	990	14
4	Wdbc	569	30
5	Winequality-Red	1599	12
6	Winequality-White	4898	12
7	Waveform	5000	22
8	Sensor Readings	5456	24
9	Letter Recognition	20000	16
10	Sgemm Gpu	241600	17
11	Kdd Cup	494020	37
12	Fma	106574	518

Table 4
The skill contexts.

No	Data sets	$\|I\|$	$\|T\|$
1	Zoo	101	17
2	Indian Liver Patient	583	9
3	Vowel-Context	990	14
4	Wdbc	569	30
5	Winequality-Red	1599	12
6	Winequality-White	4898	12
7	Waveform	5000	22
8	Sensor Readings	5456	24
9	Letter Recognition	20000	16
10	Sgemm Gpu	241600	17
11	Kdd Cup	494020	37
12	Fma	100000	1000

and the learning paths for skill, rather than the choice of learning paths for skill in this paper.

4.3. Transformation of information granules based on a skill multimap

Let (I, T, υ) be a skill multimap, and $r = \prod_{p \in I} |\upsilon(p)|$. Then the skill multimap (I, T, υ) can be decomposed into r skill contexts (I, T, R_k) $(k = 1, 2, \ldots, r)$. The transformation between information granules can be done in each skill context (I, T, R_k) $(k = 1, 2, \ldots, r)$. Through CCLP-1 and CCLP-2, sufficient and necessary competence-based information granules are formed from a general information granule in each (I, T, R_k), then they are fused according to Equation (D2) of Definition 2.6. Algorithm 2 is shown as the concept cognitive learning process of the Cb-CCLM based on a skill multimap. In Algorithm 2, let $\mathcal{N}_k$ and $\mathcal{S}_k$ be a necessary and a sufficient competence-based information granules set in the skill context (I, T, R_k) $(k = 1, 2, \ldots, r)$, respectively. The flow diagram of Algorithm 2 is Fig. 8. By the computational complexity analysis of Algorithm 1, the computational complexity of Algorithm 2 is $O(r(|I||T| + |I| + |T|))$.

Example 4.3. Let $p_0 = \{4\}, s_0 = \{r\}$ in Example 2.3. In the four skill contexts, there are: $4^\diamond \supsetneq \{r\}$ and $4^\diamond \subsetneq \{r\}$, thus $(4, r)$ is an inconsistent information granule in the four skill contexts. By CCLP-1, $(4, r)$ is transformed into necessary competence-based information granules $(\emptyset, \emptyset), (\emptyset, t), (\emptyset, r), (34, tr), (3, tr), (4, tr)$, which are transformed to sufficient and necessary competence-based information granules $(\emptyset, \emptyset), (4, t), (3, r), (34, tr), (234, tr)$. Through Equation (D2) of Definition 2.6, the sufficient and necessary information granules $(\emptyset, \emptyset), (3, r), (4, t), (234, tr)$ are obtained. Through CCLP-2, $(4, r)$ is transformed to sufficient competence-based information granules $(4, \emptyset), (34, r)$, which are converted to sufficient and necessary competence-based information granules $(\emptyset, \emptyset)$, $(4, t)$, $(3, r)$, $(34, tr)$, $(234, tr)$. Finally, the sufficient and necessary competence-based information granules $(\emptyset, \emptyset), (4, t), (3, r), (234, tr)$ are obtained by Equation (D2).

For another information granule $(34, t\upsilon)$ in Example 2.3, through CCLP-1, we can finally get the sufficient and necessary competence-based information granules $(4, t), (234, tr), (1234, tr\upsilon)$. By CCLP-2, the sufficient and necessary competence-based information granules $(4, t), (234, tr)$ can ultimately be obtained. Thus, through CCLP-1 or CCLP-2, an inconsistent information granule can be transformed into some sufficient and necessary competence-based information granules (i.e., Cb-concepts). The number of the Cb-concepts generated by CCLP-1 is not less than that obtained by CCLP-2.

Table 5
Number of the obtained Cb-concepts and running times of Algorithm 1.

No.	Number of items	CCLP-1	CCLP-2	Total number	Running time(s)	No.	Number of items	CCLP-1	CCLP-2	Total number	Running time(s)
	1 − 20(20%)	4	4	4	0.00432		1 − 1000(20%)	4	4	4	0.0272
	1 − 40(40%)	3	2	3	0.00428		1 − 2000(40%)	4	4	4	0.0404
1	1 − 60(60%)	5	4	5	0.00476	7	1 − 3000(60%)	4	4	4	0.0506
	1 − 80(80%)	4	4	4	0.00497		1 − 4000(80%)	4	4	4	0.0561
	1 − 101(100%)	6	4	6	0.00681		1 − 5000(100%)	4	4	4	0.0615
	1 − 117(20%)	4	4	4	0.0083		1 − 1091(20%)	4	4	4	0.0352
	1 − 234(40%)	4	4	4	0.0101		1 − 2182(40%)	4	4	4	0.0537
2	1 − 351(60%)	4	4	4	0.0121	8	1 − 3273(60%)	4	4	4	0.0553
	1 − 468(80%)	5	4	5	0.0131		1 − 4364(80%)	4	4	4	0.0610
	1 − 583(100%)	5	4	5	0.0144		1 − 5456(100%)	4	4	4	0.1044
	1 − 198(20%)	3	2	3	0.0074		1 − 4000(20%)	5	4	5	0.0592
	1 − 396(40%)	3	2	3	0.0098		1 − 8000(40%)	5	4	5	0.1142
3	1 − 594(60%)	3	2	3	0.0124	9	1 − 12000(60%)	5	4	5	0.1647
	1 − 792(80%)	3	2	3	0.0140		1 − 16000(80%)	5	4	5	0.2238
	1 − 990(100%)	3	2	3	0.0145		1 − 20000(100%)	5	4	5	0.2400
	1 − 114(20%)	4	4	4	0.0148		1 − 48320(20%)	4	4	4	15.3029
	1 − 228(40%)	5	4	5	0.0162		1 − 96640(40%)	5	4	5	15.5732
4	1 − 342(60%)	5	4	5	0.0180	10	1 − 144960(60%)	5	4	5	15.9073
	1 − 456(80%)	5	4	5	0.0188		1 − 193280(80%)	5	4	5	16.2036
	1 − 569(100%)	6	4	6	0.0208		1 − 241600(100%)	4	4	4	16.6455
	1 − 320(20%)	5	4	5	0.0109		1 − 98804(20%)	5	4	5	14.3099
	1 − 640(40%)	5	4	5	0.0133		1 − 197608(40%)	6	4	6	15.4737
5	1 − 960(60%)	4	4	4	0.0164	11	1 − 296412(60%)	4	4	4	15.9403
	1 − 1280(80%)	4	4	4	0.0192		1 − 395216(80%)	4	4	4	16.6587
	1 − 1599(100%)	5	4	5	0.0250		1 − 494020(100%)	5	4	5	17.4045
	1 − 980(20%)	4	4	4	0.0163		1 − 20000(20%)	5	4	5	8.6945
	1 − 1960(40%)	4	4	4	0.0287		1 − 40000(40%)	5	4	5	11.3038
6	1 − 2940(60%)	4	4	4	0.0346	12	1 − 60000(60%)	4	4	4	11.2823
	1 − 3920(80%)	4	4	4	0.0384		1 − 80000(80%)	5	4	5	13.8349
	1 − 4898(100%)	4	4	4	0.0487		1 − 100000(100%)	4	4	4	16.4373

5. Experiments of transformation between information granules in a Cb-CCLM

In this section, some experiments have been done to assess the Cb-CCLM. The computer with a Window10 operating system owns an Intel(R) Core(TM) i7-9750H CPU @ 2.60 GHz processor, 24 GB RAM. To obtain the validity and feasibility of the Cb-CCLM, the experiments are performed by using Matlab in some UCI data sets, which are underlined in Table 3. To get the skill contexts, each data is dealt with the average value in its corresponding column in each data sets. Then, through the data preprocessing, these data sets are converted into skill contexts shown in Table 4. Especially, Algorithm 1 plays a central role in this article. Thus, Algorithm 1 is tested in these data sets. To test the time efficiency and the number of the obtained Cb-concepts (i.e., sufficient and necessary competence-based information granules), 20%, 40%, 60%, 80% and 100% items of these data sets are took as the initial items. The number of Cb-concepts obtained by CCLP-1 or CCLP-2, the total number of the obtained Cb-concepts and the running time are shown in Table 5 and Figs. 9–11, respectively. From Table 5, Figs. 9 and 11, we can see that the Algorithm 1 proposed in this paper is effective and feasible.

From Table 5, we can find that: For any information granule in each data set, several sufficient and necessary competence-based information granules (i.e., Cb-concepts) can be obtained through CCLP-1 or CCLP-2. And the number of Cb-concepts formed through CCLP-1 is not less than that obtained through CCLP-2. The p and s of the information granule (p, s) constructed through Proposition 4.5 (S1, S2), are related to the information granule generated from Proposition 4.4 (N1, N2, N5, N6) with the same initial information granule. Thus, we can deduce that the number of sufficient and necessary competence-based information granules formed through CCLP-1 (Propositions 4.4 and 4.6) is not less than that obtained through CCLP-2 (Propositions 4.5 and 4.7). Therefore, the results of Table 5 are confirmed with Propositions 4.4–4.7. From Figs. 9–10, we can obtain that the number of the obtained Cb-concepts will be affected with the initial information granule and the initial items number of each data set. Sometimes, the same Cb-concepts may be occasionally generated from different information granule, it is due to the structure of concept lattice, redundant skills and items. From Table 5 and Figs. 9–11, it can be seen that the number of the Cb-concepts is much less than 16.

Fig. 11 shows that different data sets pay different running time to the number of initial items, skills and the obtained Cb-concepts. From Fig. 11, we can obtain that: (1) When the number of initial items is fixed, the more Cb-concepts are generated, the running time is a little larger. (2) When the number of Cb-concepts is given, the running time increases as the number of initial items increases.

To test the influence of the number of skills on the running time, we compare two groups of data sets, one group is Indian Liver Patient and Wdbc, the other group is Winequality-White, Waveform and Sensor Readings. The results are shown in Fig. 12. From Fig. 12, we can get that when the number of initial items and the obtained Cb-concepts are given, the running time increases with the number of skills. In general, the knowledge structure with more skills is more complex.

Furthermore, to test the performance of the proposed algorithm 1, we compare it with two classical models, including the two-ways cognitive learning model (TCLM) [3] and the cognitive learning model via granular concepts (GCLM) [10]. The number of the obtained concepts and the running times are shown in Table 6. From Table 6, we can see that the running times of Cb-CCLM are better than GCLM especially in big data, and they have no advantages compared with the TCLM. But compared with TCLM and GCLM, the Cb-CCLM has the advantage of dealing with the concept-cognitive learning from the perspective of competences, and it is good at describing the relationship between knowledge and skills.

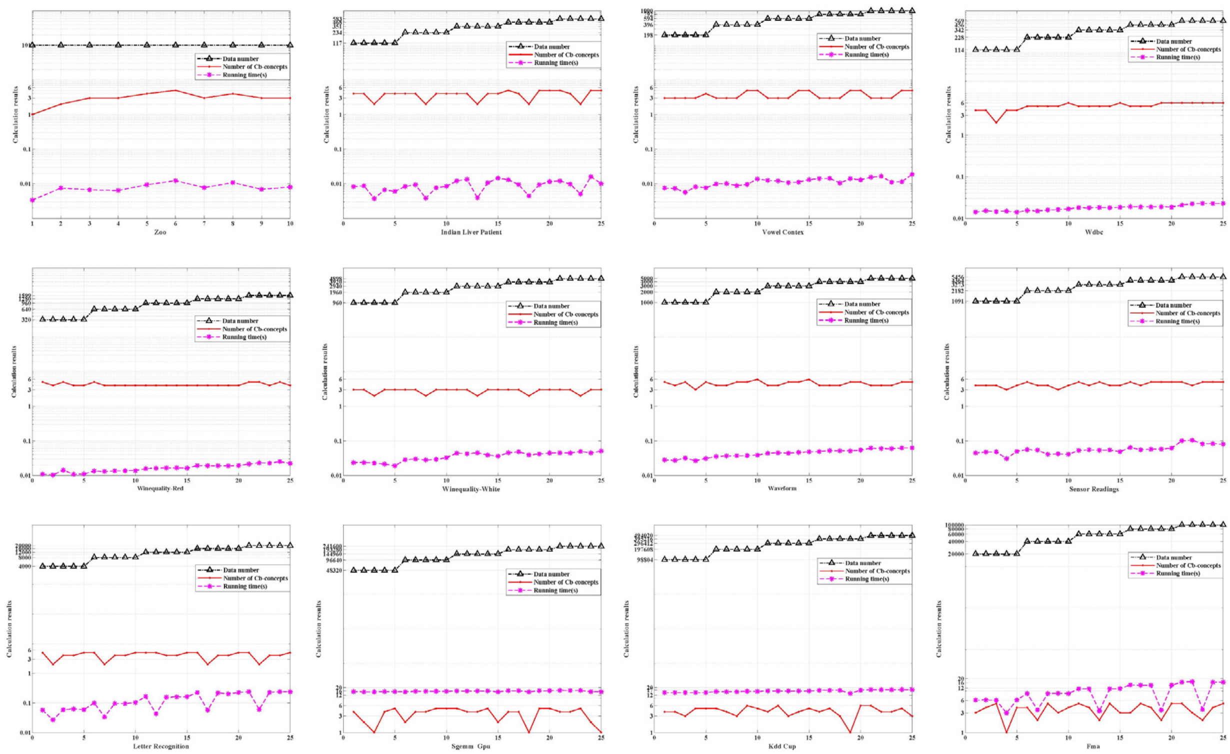

Fig. 9. Number of the obtained Cb-concepts and running times.

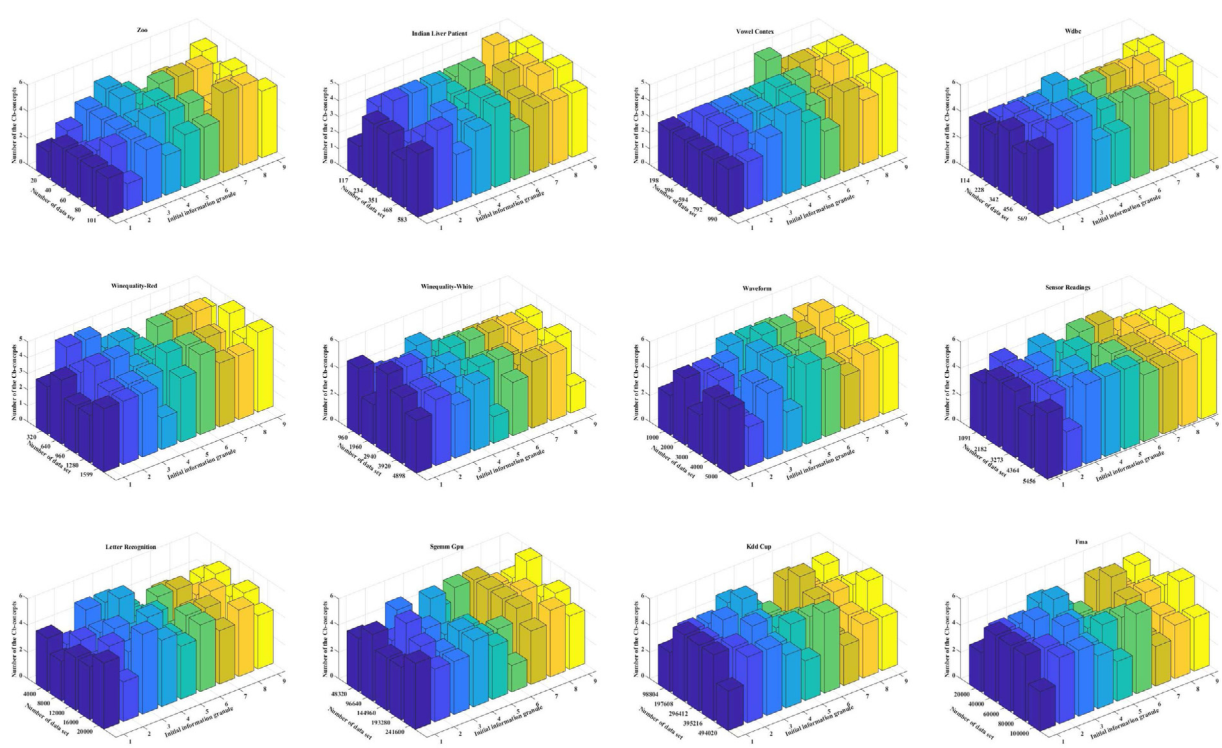

Fig. 10. Number of the obtained Cb-concepts varies with initial information granule and initial items number.

6. Conclusions

The Cb-CCLM provides a new method for the concept-cognitive learning through skills, and makes a useful extension of the concept-cognitive learning. One needs to acquire not only knowledge and its commonalities, but also the skills beyond to solve different problems. The Cb-CCLM can actually study the transformation relationship between skills and knowledge from the perspective of competences. By utilizing the structure of information granules, and the transformation method between information granules, one can learn and obtain several valuable sufficient and necessary competence-based information granules (i.e., Cb-concepts) from a general information granule though

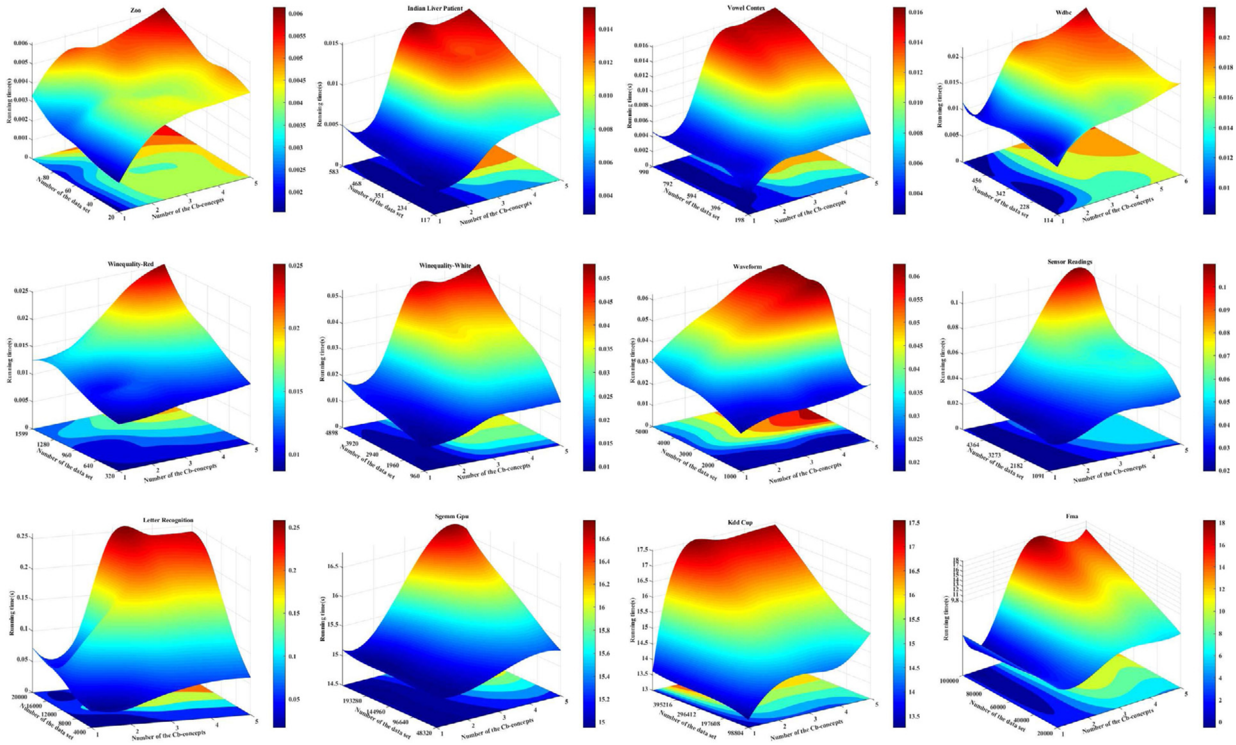

Fig. 11. Running time varies with the number of initial items and the obtained Cb-concepts.

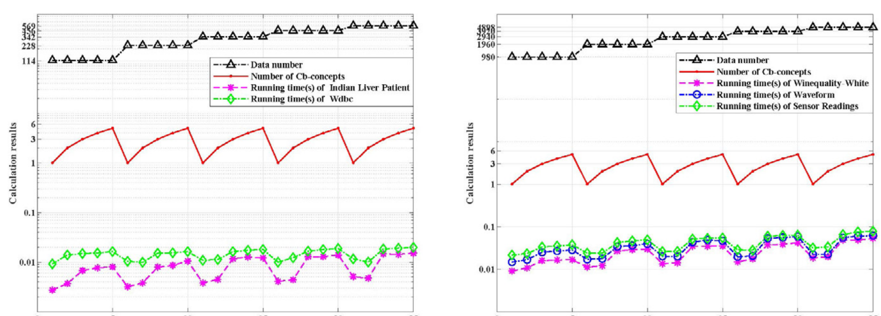

Fig. 12. Running time varies with the number of skills.

Table 6
Number of the obtained concepts and running times of the comparative algorithms.

No	Data sets	$\|I\|$	$\|T\|$	Number of the obtained concepts			Running times(s)		
				TCLM	GCLM	Cb-CCLM	TCLM	GCLM	Cb-CCLM
1	Zoo	101	17	4	2	4	0.004035	0.006153	0.006048
2	Indian Liver Patient	583	9	3	1	3	0.006203	0.026241	0.020371
3	Vowel-Context	990	14	5	2	5	0.007631	0.076994	0.012492
4	Wdbc	569	30	4	2	4	0.012816	0.044867	0.022334
5	Winequality-Red	1599	12	5	2	5	0.012040	0.191601	0.021461
6	Winequality-White	4898	12	6	2	5	0.030312	1.730886	0.058767
7	Waveform	5000	22	5	2	5	0.049527	2.220761	0.063262
8	Sensor Readings	5456	24	4	2	4	0.058344	2.551559	0.076891
9	Letter Recognition	20000	16	5	2	4	0.188423	30.65526	0.231778
10	Sgemm Gpu	241600	17	5	2	5	15.369847	1795.461813	16.608517
11	Kdd Cup	494020	37	4	2	4	14.375400	7691.206231	17.354660
12	Fma	100000	1000	5	2	5	6.497687	8953.495554	16.583840

CCLP-1 or CCLP-2. The obtained Cb-concepts mean that one can learn the least amount of skills for solving the most amount of items. It fits the expectation of cognitive learning from the perspective of economic cost. And the obtained Cb-concepts are conducive for one to get the learning paths for skill and realize the personalized learning. In additional, the experimental results show that the transformation between information granules is feasible and effective even in big data. However, it is impossible to determine which Cb-concept is the best one of the obtained Cb-concepts in the Cb-CCLM. Thus, we will study to enhance the result of concept-cognitive learning through educational and learning rules. In additional, skill proficiency [43] will affect the mastery of knowledge, therefore, we will try to connect the concept-cognitive learning with skill proficiency.

CRediT authorship contribution statement

Xiaoxian Xie: Data curation, Methodology, Software, Visualization, Writing – original draft, Writing – review & editing. **Weihua Xu:** Conceptualization, Funding acquisition, Investigation, Methodology, Project administration, Supervision, Validation. **Jinjin Li:** Conceptualization, Funding acquisition, Investigation, Methodology, Project administration, Supervision, Validation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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