A Novel Dynamic Fusion Approach Using Information Entropy for Interval-Valued Ordered Datasets

Weihua Xu, Yanzhou Pan, Xiuwei Chen, Weiping Ding, and Yuhua Qian

Abstract—Information fusion is capable of fusing and transforming information originated from multiple sources into an integrated representation. As an important representative of information, interval-valued ordered data aims at characterizing inaccurate and ambiguous information. However, existing methods are inappropriate when applied to the multi-source fusion for them. In addition, it is inevitable that in daily-life the sources and attributes of multi-source information systems may update at the same time. Hence there is a need to fuse data as efficiently as possible. Inspired by these deficiencies, we pay attention to the effective and efficient fusion of multi-source interval-valued ordered data in varieties of cases. First, the concepts of fuzzy dominance relation and dominance classes based on it are put forward between any two samples of interval-valued ordered information systems. Second, we define the fuzzy dominating and dominated conditional entropy. Then the fusion model is established and it is multi-source interval-valued ordered data oriented. Furthermore, it is true that there are numerous real-life applications related to concurrent change of both sources and attributes. Consequently, we design four incremental mechanisms and algorithms for fusing multi-source interval-valued data on the ground of the static condition. Eventually, a series of experiments is carried out on twelve datasets to verify that the proposed fusion approach outperforms other comparative methods on efficacy. Meanwhile, our incremental fusion algorithms are efficient compared with the static one for updating multi-source interval-valued data when sources and attributes are in the simultaneous variation.

Index Terms—Dynamic fusion, information entropy, interval-valued ordered data, multi-source information system

1 Introduction

Interval-valued data can be employed to effectively characterize uncertainty phenomena widely existing in human society, such as the fluctuation of stock price [1], the range of physical indicators [2] and the variation in temperature [3]. Oceans of researches about interval-valued data have been investigated on decision analysis [4], cluster [3], and data mining [5]. What's more, Dai et al. investigated uncertainty measures for incomplete interval-valued information systems via defining an $\alpha-$weak similarity relation [6]. In multi-criteria decision systems, a preference-ordered relation is provided for interval-valued objects based on prior knowledge of decision-makers, which are called interval-valued ordered decision systems [7]. For interval-valued ordered data, massive studies have been conducted regarding rough approximations using incremental methods [8], [9] and feature selection [10], [11]. The aforementioned researches all concentrate on a single information system. However, with the fast advancement of big data technologies, aggregating and analyzing data from numerous sources or sensors may significantly upgrade the efficacy of decision-making. For instance, we may gather weather data from varieties of sensors to increase the precision of weather forecasting. And the existing methods on multi-source information fusion, such as [12], [13], [14], are unsuitable for the interval ordered data. Thus, it is necessary to promote the approach for using multi-source interval-valued ordered data to enhance data mining effectiveness.

As a useful method to deal with multi-source data, information fusion can transform and fuse information collected from multiple sources to compose an integrated representation. Two effective tools are often used in information fusion theories, i.e., rough set theory (RST) [15] and information entropy. For one thing, RST approximates concept sets via the approximations operators, which can effectively quantify uncertainty [16], [17], remove redundant attributes [18], and extract rules [19]. There are also some extended researches on rough sets with soft sets such as [20], [21]. [22] provided a brand new perspective into rough set theory from the angle of $\mathcal{N}-$ soft sets. And based on RST, plenty of scholars have come up with developed models on information fusion [23], [24], [25]. [26] offered a general overview of information fusion based on RST in five respects, i.e., objects, attributes,

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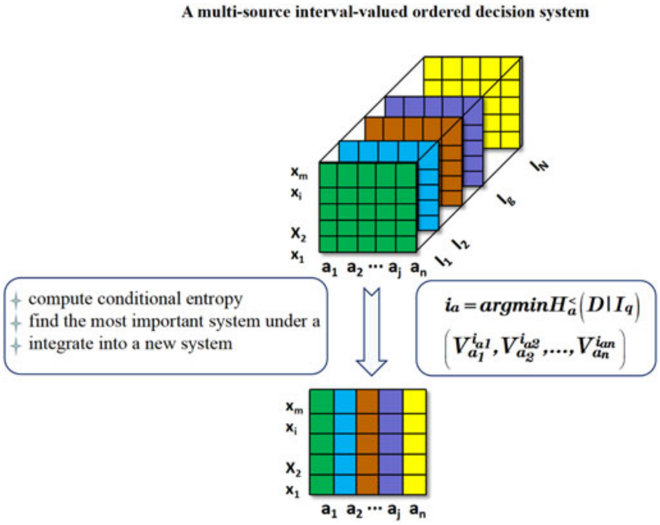

Fig. 1. The fusion process of the proposed method.

rough approximations, attribute reduction and decision making. For example, Greco et al. put forward dominance rough sets for ordered data to show the preference relation among data [27]. However, this traditional approach only includes boolean relations, which are weak in quantitatively expressing the degree to which one object is superior to another. Then Hu et al. combined fuzzy set theory (FST) [28], used to evaluate the degree to which a sample belongs to a set, with RST and proposed fuzzy dominance rough set [29].

For another thing, information entropy was initially presented by Shannon as a mathematical instrument for uncertainty measurement and nowadays it has been widely used in information processing [30]. Massive entropy-based methods have been developed and applied into plenty of fields, such as [31] and [32]. Based on [28], the author defined fuzzy dominated and dominating conditional entropy in [33]. As information fusion research goes on, many information fusion methods involving RST and information entropy have been put forward. In fuzzy multi-source incomplete systems, Xu et al. used conditional entropy to determine the most essential sources for an attribute [34]. Later, for multi-label information systems, Qian et al. presented a novel fusion framework by utilizing information entropy [35].

However, the above techniques cannot be directly applied to fuse interval-valued ordered data. And for more specific descriptions of dominance relations between samples, we introduce a fuzzy dominance conditional entropy for interval-valued data in multi-source fusion. Via utilizing a novel conditional entropy of interval-valued ordered data, this article presents an approach for multi-source fusion of interval-valued ordered decision systems (Ms-IVODS), which is shown in Fig. 1. First, we give the concept of the fuzzy dominance relation between two interval-valued ordered samples. Then, fuzzy dominating and dominated classes of each sample are defined. On that basis, fuzzy dominating and dominated conditional entropy can be calculated. For an attribute, the smaller the conditional entropy is, the more significant the system can be. That means we need to compute the conditional entropy respectively under different systems for each attribute, and try to seek out the most important system for the attribute. Finally, we integrate the attribute values in those systems to a new system as the fusion result.

Another incentive of the paper is to better adapt to the era of big data. Undoubtedly, it is time-consuming for dynamic data to employ static methods. In order to efficiently deal with updating data, numerous researchers have explored such as [36], [37]. In [38], the authors depicted the future direction of multi-source information fusion and incremental learning still showed great potentiality. Nowadays, numerous incremental updating methods, related to the dynamic variation of objects [39], attributes [40] or object sets [41], have been developed in the context of a single information system. These methods are all primarily used to access dynamic single-source data. As a generalization of single source data, multi-source data can contain more information. Consequently it is meaningful and necessary to explore how to fuse dynamic multi-source data. However, existing researches, for instance, [42], [43], [44] just have kept an eye on the static multi-source environment, without considering the variation of information systems. There is no denying that static fusion methods will spend much more time acquiring the updated fusion results for dynamic data. In order to speed up the access time, many dynamic fusion approaches have been put forward with the change of information. Huang et al. developed a method for fusing interval-valued data dynamically as the number of information sources changes continually [45]. Zhu et al. proposed an approach for progressive fusion of fuzzy and uncertain data [46].

The above researches just paid attention to one single variant, while our study concentrates on the dynamic multi-source interval-valued ordered data whose sources and attributes change in the meantime. In real life, the number of sources and attributes may change at the same time. For instance, when predicting weather condition of a city, we can gather various information such as temperature, humidity, and wind speed from weather sensors all over the city. However, with the improvement of the science and technology, wind speed is found to be unimportant for weather forecast, so the data concerning wind speed will not be collected in order to save acquisition cost of information. At the same time, obsolete weather sensors need to be removed so as not to affect the accuracy of prediction. In this case, the sources and attributes change simultaneously, which means it is necessary to study dynamic fusion approaches of Ms-IVODS under this condition. There is no doubt that many single-updating fusion approaches in existence, such as [45] and [46] are not suitable for fusing Ms-IVODS when information sources and attributes change together. In this paper, four dynamic fusion methods are presented based on four cases, which are shown in Fig. 2.

The contributions are summarized as follows: 1) We establish a fuzzy dominance relation for any two interval-valued ordered data. Furthermore, a novel conditional entropy is proposed to quantify the significance of sources to attributes based on this relationship.

2) This paper presents an entropy-based framework for fusion of multi-source interval-valued ordered data. For each attribute, we separately compute conditional entropy in each system and find out the system with the minimum

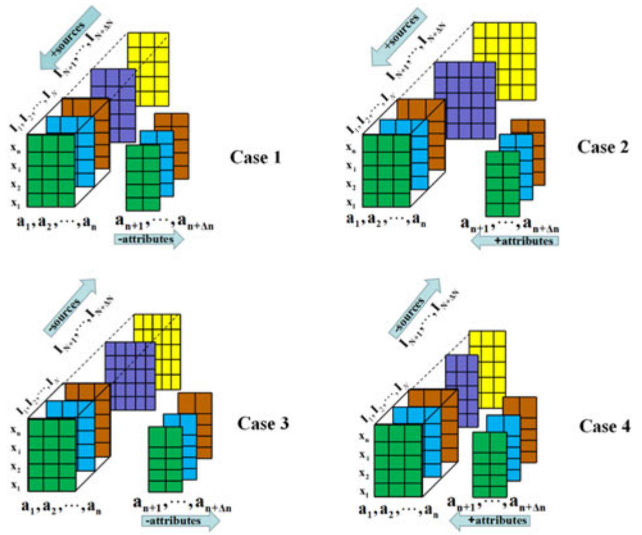

Fig. 2. Four changing cases of Ms-IVODS.

outcome. Then in accordance with different attributes, we integrate all the corresponding values of the attribute into a new system as the fusion result. On the ground of the static method, four dynamic updating mechanisms are developed with the variation of sources and attributes.

3) Experiments on twelve datasets from UCI are used to verify that the proposed algorithms contribute to upgrade the performance of fusion, which reflects on the classification accuracy. Additionally, it is illustrated that the incremental mechanisms are conducive to significantly reduce calculation time when both sources and attributes change concurrently.

From the above, much effort has been dedicated to obtaining a unified representation of multi-source interval-valued ordered data and updating the fusion results quickly. Nevertheless, the improved algorithms still suffer from the following deficiencies and challenges.

1) The entropy-based fusion algorithm needs to compute the conditional entropy of each attribute under every information system, and its time complexity can be calculated through $O(n \times |AT| \times |U|^2 + |AT|)$. However, the time complexity of the state-of-the-art fusion approach [41] can be required through $O(|U| \times |AT| \times (2n^2 + n\log_2(n) + 4n))$. When there are just fewer sources, the computation time of our method is larger instead.
2) The presented approaches can speed up access time when sources and attributes change at the same time. Nevertheless, it is common that the sources, attributes, and samples are always in the simultaneous variation, with which our methods are less-suitable to cope.
3) There is still room for improvement in practical applications. For instance, [47] enlightens us whether we can apply the the proposed dynamic fusion approach to linguistic terms with weakened hedges. We will further explore it in the future work.

The remainder of this paper is arranged as follows. In Section 2, several fundamental concepts are introduced. Section 3 puts forward a static fusion approach on the basis of

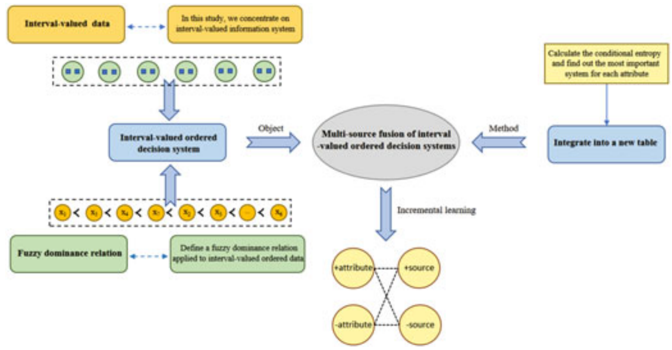

Fig. 3. The brief framework of the study.

conditional entropy for Ms-IVODS, furthermore, four dynamic techniques in Section 4. And grounded on them, we develop four incremental algorithms and compare the four mechanisms with the static one on time complexity in Section 5. Section 6 presents the results of the experimental analysis. Finally, Section VII draws the conclusion of this paper. And the brief framework of the paper can be seen in Fig. 3.

2 Preliminaries

This section introduces some basic concepts, i.e., ordered decision systems (ODS), fuzzy set, conditional entropy, and multi-source interval-valued ordered decision systems (Ms-IVODS).

2.1 Ordered Decision Systems

We first introduce a basic concept called decision systems. Let $IS = (U, AT, V\_AT, f)$ be an information system [15], where U is the samples set; AT is the condition attributes set; V_AT is the domain of AT; f: $U \times AT \to V$ is an information function. In particular, a decision system can be denoted as $DS = IS \cup \{D, V\_D, f\_D\}$, where D is the decision attribute set, V_D is the domain of D and f_D: $U \times D \to V\_D$ is an information function. For any $B \subseteq AT$, a relation $R_B \subseteq U \times U$ can be defined. And for any $x \in U$, the relation class of x is defined as $[\mathrm{x}]^{R_B} = \{y \in U | \mathrm{x} R_B y\}$. For any $X \subseteq U$, the lower and upper approximations of X are defined by

$$\underline{R_B}(X) = \left\{x \in U \middle| [\mathrm{x}]^{R_B} \subseteq X\right\},$$
$$\overline{R_B}(X) = \left\{x \in U \middle| [\mathrm{x}]^{R_B} \cap X \neq \emptyset\right\}.$$

Let $U/D = \{Y_1, Y_2, \ldots, Y_m\}$ be a partition of the universe U based on the decision attributes set D. For a DS, the approximation precision(AP) and approximation quality(AQ) under R_B are defined as [48], [49]:

$$AP_{R_B}(U/D) = \frac{\sum_{i=1}^{m} \left|\underline{R_B}(Y_i)\right|}{\sum_{i=1}^{m} \left|\overline{R_B}(Y_i)\right|},$$
$$AQ_{R_B}(U/D) = \frac{\sum_{i=1}^{m} \left|\underline{R_B}(Y_i)\right|}{|U|}.$$

And we are going to give a detailed introduction of AP and AQ in the experimental part. Now, we give the definition of ordered decision systems. A decision system can be called an ordered decision system (ODS) [50] if for any $a \in$

TABLE 1
Medical Report in Hospital 1- I_1

U	a_1	a_2	a_3	a_4	a_5	a_6
x_1	$[0.90, 1.30]$	$[0.03, 0.06]$	$[1.10, 1.20]$	$[0.64, 1.00]$	$[0.40, 1.70]$	$[0.65, 0.85]$
x_2	$[0.83, 1.20]$	$[0.04, 0.10]$	$[1.20, 1.60]$	$[0.74, 1.10]$	$[0.75, 2.10]$	$[0.62, 0.82]$
x_3	$[0.76, 1.10]$	0	$[1.00, 1.10]$	$[1.10, 1.70]$	$[0.62, 0.79]$	$[0.73, 0.92]$
x_4	$[0.88, 1.20]$	$[0.02, 0.08]$	$[1.00, 1.10]$	$[0.56, 0.92]$	$[0.93, 2.40]$	$[0.62, 0.82]$
x_5	$[0.89, 1.20]$	$[0.04, 0.10]$	$[1.00, 2.80]$	$[0.76, 1.10]$	$[1.30, 2.60]$	$[0.41, 0.69]$
x_6	$[0.88, 1.90]$	$[0.06, 0.14]$	$[1.90, 2.80]$	$[0.75, 1.50]$	$[0.63, 0.82]$	$[0.26, 0.57]$
x_7	$[0.86, 1.80]$	$[0.04, 0.08]$	$[1.60, 2.50]$	$[0.78, 1.40]$	$[0.72, 0.91]$	$[0.28, 0.56]$
x_8	$[1.10, 2.10]$	$[0.09, 0.18]$	$[2.10, 2.90]$	$[0.95, 1.50]$	$[0.61, 0.79]$	0
x_9	$[0.83, 1.40]$	$[0.11, 0.20]$	$[2.10, 2.50]$	$[0.84, 1.40]$	$[0.66, 0.85]$	$[0.72, 0.82]$
x_{10}	$[0.99, 1.90]$	$[0.08, 0.16]$	$[1.60, 2.50]$	0	$[1.00, 2.40]$	$[0.37, 0.65]$

AT, V_a is completely pre-ordered by the relation $\preceq_a$: $x \preceq_a y \Leftrightarrow f(x,a) \leq f(y,a)$, which means y is at least as good as x, called an increasing preference or $x \preceq_a y \Leftrightarrow f(x,a) \geq f(y,a)$, which means x is at least as good as y, called an decreasing preference.

2.2 Fuzzy Set Theory

According to [29], a fuzzy set $\widetilde{X}$ on U can be denoted as $\widetilde{X} = \frac{\widetilde{X}(x_1)}{x_1} + \frac{\widetilde{X}(x_2)}{x_2} + \ldots + \frac{\widetilde{X}(x_n)}{x_n}$, where $\widetilde{X} \in [0,1]$ is the membership function. For any two fuzzy sets, we say $\widetilde{X} \subseteq \widetilde{Y}$, if $\forall x \in U$, $\widetilde{X}(x) \leq \widetilde{Y}(x)$. Furthermore, some basic operational rules are defined as [29] $\widetilde{X} \cup \widetilde{Y} = \frac{max(\widetilde{X}(x_1),\widetilde{Y}(x_1))}{x_1} + \frac{max(\widetilde{X}(x_1),\widetilde{Y}(x_2))}{x_2} + \ldots + \frac{max(\widetilde{X}(x_1),\widetilde{Y}(x_n))}{x_n}$, $\widetilde{X} \cap \widetilde{Y} = \frac{min(\widetilde{X}(x_1),\widetilde{Y}(x_1))}{x_1} + \frac{min(\widetilde{X}(x_1),\widetilde{Y}(x_2))}{x_2} + \ldots + \frac{min(\widetilde{X}(x_1),\widetilde{Y}(x_n))}{x_n}$, $\widetilde{X}^c = \frac{1-\widetilde{X}(x_1)}{x_1} + \frac{1-\widetilde{X}(x_2)}{x_2} + \ldots + \frac{1-\widetilde{X}(x_n)}{x_n}$, $|\widetilde{X}| = \sum_{x \in U} \widetilde{X}(x)$.

2.3 Multi-Source Interval-Valued Ordered Decision Systems

An interval-valued information system(IVIS) [4] can be denoted as $IVIS = (U, AT, V\_AT, f)$, where the information function $f(x,a) = [f^L(x,a), f^U(x,a)]$ and for any $a \in AT$, $x \in U$, $f^L(x,a)$ and $f^U(x,a)$ denote the lower and upper endpoints of interval.

An IVIS is referred to as an interval-valued ordered information system(IVOIS) [7] if for any $a \in AT$, V_a is completely pre-ordered by the relation $\preceq_a : x \preceq_a y \Leftrightarrow f^L(x,a) \leq f^L(y,a) \wedge f^U(x,a) \leq f^U(y,a)$, called an increasing preference or $x \preceq_a y \Leftrightarrow f^L(x,a) \geq f^L(y,a) \wedge f^U(x,a) \geq f^U(y,a)$, called an decreasing preference. Specially, an interval-valued ordered decision system(IVODS) can be denoted as $IVODS = (U, AT_i, V\_AT_i, f_i, D, V\_D, f\_D)$.

A multi-source interval-valued ordered information system(Ms-IVOIS) can be expressed as $Ms-IVOIS = \{I_i | I_i = (U, AT_i, V\_AT_i, f_i), i = 1, 2, \ldots, n\}$, where I_i represents i-th IVOIS of Ms-IVOIS. Specially, a multi-source interval-valued ordered decision system(Ms-IVODS) can be denoted as $Ms-IVODS = \{Ms-IVOIS \cup \{D, V\_D, f\_D\}\}$.

Example 1. In this example, we are going to talk about a medical examination issue. 10 patients have thyroid examinations at four different hospitals. 6 relative checks are made for each patient and they are marked by a_1 to a_6, which respectively stands for six corresponding indices: "FT3", "FT4", "TSH", "TPOAb", "TGAb", "TT3". Tables 1, 2, 3, and 4 are examination results performed at four hospitals. Meanwhile in each table, D represents conditions of patients and is divided into three categories, where 3 means hyperthyroidism, 2 indicates a sound condition and 1 implies hypothyroidism. And suppose $V\_D = \{3,1,2,1,3,2,3,3,2,1\}$. The four information tables and D make up an $Ms-IVODS = \{Ms-IVOIS \cup \{D, V\_D, f\_D\}\}$, where $Ms-IVOIS = \{I_i | I_i = (U, AT_i, \ V\_AT_i, f_i), i \ = 1,2,3,4\}$, $U = \{x_1, x_2, \ldots, x_{10}\}$, and $AT_i = \{a_1, a_2, a_3, a_4, a_5, a_6\}$.

3 Information Fusion of Ms-IVODS With Information Entropy

Fusion of data from many sources can generate a complete and unified representation which is conducive to enhance

TABLE 2
Medical Report in Hospital 2- I_2

U	a_1	a_2	a_3	a_4	a_5	a_6
x_1	$[1.10, 1.40]$	$[0.03, 0.07]$	$[1.10, 1.20]$	$[0.67, 1.10]$	$[0.43, 1.80]$	$[0.69, 0.90]$
x_2	$[1.00, 1.30]$	$[0.04, 0.11]$	$[1.20, 1.70]$	$[0.78, 1.20]$	$[0.79, 2.20]$	$[0.66, 0.86]$
x_3	$[0.94, 1.20]$	0	$[1.10, 1.20]$	$[1.20, 1.80]$	$[0.65, 0.83]$	$[0.77, 0.97]$
x_4	$[1.10, 1.30]$	$[0.02, 0.09]$	$[1.10, 1.20]$	$[0.60, 0.97]$	$[0.98, 2.50]$	$[0.65, 0.86]$
x_5	$[1.10, 1.30]$	$[0.04, 0.11]$	$[1.10, 2.90]$	$[0.80, 1.20]$	$[1.40, 2.80]$	$[0.43, 0.72]$
x_6	$[1.10, 2.10]$	$[0.06, 0.16]$	$[2.00, 2.90]$	$[0.79, 1.60]$	$[0.67, 0.86]$	$[0.27, 0.60]$
x_7	$[1.10, 1.90]$	$[0.04, 0.09]$	$[1.70, 2.60]$	$[0.83, 1.50]$	$[0.76, 0.96]$	$[0.29, 0.59]$
x_8	$[1.40, 2.30]$	$[0.10, 0.20]$	$[2.20, 3.10]$	$[1.00, 1.60]$	$[0.64, 0.83]$	0
x_9	$[1.00, 1.50]$	$[0.12, 0.22]$	$[2.20, 2.26]$	$[0.88, 1.50]$	$[0.70, 0.89]$	$[0.76, 0.86]$
x_{10}	$[1.20, 2.10]$	$[0.08, 0.18]$	$[1.70, 2.60]$	0	$[1.10, 2.50]$	$[0.39, 0.69]$

TABLE 3
Medical Report in Hospital 3- I_3

U	a_1	a_2	a_3	a_4	a_5	a_6
x_1	$[0.95, 1.30]$	$[0.03, 0.07]$	$[1.00, 1.10]$	$[0.62, 0.98]$	$[0.39, 1.70]$	$[0.63, 0.83]$
x_2	$[0.88, 1.20]$	$[0.04, 0.10]$	$[1.10, 1.60]$	$[0.72, 1.10]$	$[0.73, 2.00]$	$[0.61, 0.80]$
x_3	$[0.81, 1.10]$	0	$[1.00, 1.10]$	$[1.10, 1.60]$	$[0.60, 0.77]$	$[0.71, 0.90]$
x_4	$[0.93, 1.20]$	$[0.02, 0.08]$	$[1.00, 1.11]$	$[0.55, 0.90]$	$[0.90, 2.40]$	$[0.60, 0.80]$
x_5	$[0.94, 1.30]$	$[0.04, 0.10]$	$[1.00, 2.70]$	$[0.74, 1.10]$	$[1.30, 2.60]$	$[0.40, 0.67]$
x_6	$[0.94, 2.00]$	$[0.06, 0.15]$	$[1.80, 2.70]$	$[0.73, 1.50]$	$[0.62, 0.80]$	$[0.25, 0.55]$
x_7	$[0.91, 1.80]$	$[0.04, 0.08]$	$[1.60, 2.40]$	$[0.76, 1.40]$	$[0.71, 0.89]$	$[0.27, 0.54]$
x_8	$[1.20, 2.20]$	$[0.09, 0.19]$	$[2.00, 2.80]$	$[0.93, 1.50]$	$[0.59, 0.77]$	0
x_9	$[0.88, 1.40]$	$[0.11, 0.20]$	$[2.00, 2.40]$	$[0.82, 1.40]$	$[0.64, 0.82]$	$[0.71, 0.80]$
x_{10}	$[1.00, 2.00]$	$[0.08, 0.17]$	$[1.60, 2.40]$	0	$[0.99, 2.30]$	$[0.36, 0.63]$

the efficacy of knowledge discovery. Additionally, information entropy is able to quantify the quantity of information and the significance of information sources. Then in this section we present a technique for Ms-IVODS fusion based on information entropy.

Definition 1. *Given an $IVOIS = (U, AT, V\_AT, f)$, where $U = \{x_1, x_2, \ldots, x_n\}$. $\forall x, y \in U$ and for any $a \in AT$, the fuzzy dominance relation between x and y under a is defined as follow:*

$$\widetilde{R_a}(x,y) = \frac{1}{1 + e^{-k\left(f^L(y,a) - f^L(x,a) + f^U(y,a) - f^U(x,a)\right)}}, \tag{1}$$

where k is a positive integer. $\widetilde{R_a}(x,y)$ reflects the fuzzy dominance relation between the objects x and y under the attribute a. What's more, it not only shows the fact that y is indeed superior to x but also evaluates the degree to which y is greater than x. Then the parameter k is introduced. The parameter k, a positive constant, is used to regulate the preference degree by users. In real-life situations, k is settled on the basis of users' preference. For any $B \subseteq AT$, we define $\widetilde{R_B}(x,y) = \min_{b \in B} \widetilde{R_b}(x,y)$, and the fuzzy dominating and dominated classes of x_i under B are defined as follow:

$$[x_i]^{\geq}_{\widetilde{R_B}} = \frac{\widetilde{R_B}(x_i,x_1)}{x_1} + \frac{\widetilde{R_B}(x_i,x_2)}{x_2} + \cdots + \frac{\widetilde{R_B}(x_i,x_n)}{x_n}, \tag{2}$$

$$[x_i]^{\leq}_{\widetilde{R_B}} = \frac{\widetilde{R_B}(x_1,x_i)}{x_1} + \frac{\widetilde{R_B}(x_2,x_i)}{x_2} + \cdots + \frac{\widetilde{R_B}(x_n,x_i)}{x_n}. \tag{3}$$

It should be noted that we set k to be 1 in this paper.

In the following, we will give the concept of entropy, which is used to assess the uncertainty in a decision system.

Definition 2. *Given an $Ms-IVODS = \{Ms-IVOIS \cup D, V\_D, f\_D\}$. For any $a \in AT$, the fuzzy dominating and dominated conditional entropy of D are defined as follow:*

$$H_a^{>}(D|I_q) = -\frac{1}{|U|}\sum_{i=1}^{n} \log \frac{\left|[x_i]^{\geq}_{\widetilde{R_a}} \cap [x_i]^{\geq}_{R_D}\right|}{\left|[x_i]^{\geq}_{\widetilde{R_a}}\right|}, \tag{4}$$

$$H_a^{<}(D|I_q) = -\frac{1}{|U|}\sum_{i=1}^{n} \log \frac{\left|[x_i]^{\leq}_{\widetilde{R_a}} \cap [x_i]^{\leq}_{R_D}\right|}{\left|[x_i]^{\leq}_{\widetilde{R_a}}\right|}, \tag{5}$$

where $[x_i]^{\geq}_{R_D} = \frac{R_D(x_i,x_1)}{x_1} + \frac{R_D(x_i,x_2)}{x_2} + \cdots + \frac{R_D(x_i,x_n)}{x_n}$, $[x_i]^{\leq}_{R_D} = \frac{R_D(x_1,x_i)}{x_1} + \frac{R_D(x_2,x_i)}{x_2} + \cdots + \frac{R_D(x_n,x_i)}{x_n}$, and

$$R_D(x,y) = \begin{cases} 1, & f_D(y,d) \geq f_D(x,d), \\ 0, & otherwise. \end{cases}$$

From (1) of Proposition 1, we can find that, $\forall x_i \in U$, the closer $\frac{|[x_i]^{\leq}_{\widetilde{R_a}} \cap [x_i]^{\leq}_{R_D}|}{|[x_i]^{\leq}_{\widetilde{R_a}}|}$ is to 1, the closer $H_a^{<}(D|I_q)$ or $H_a^{>}(D|I_q)$ is to 0, which means the less conflict between a and D. Thus, for the attribute a, the smaller the $H_a^{<}(D|I_q)$ or $H_a^{>}(D|I_q)$ is, the more important the I_q is.

Proposition 1. *It is true that the fuzzy dominating and dominated conditional entropy satisfies*

1) $H_a^{<}(D|I_q) \geq 0,\ H_a^{>}(D|I_q) \geq 0.$
2) $H_a^{<}(D|I_q) < \infty,\ H_a^{>}(D|I_q) < \infty.$
3) $H_a^{<}(D|I_q) \leq H_b^{<}(D|I_q)$ *or* $H_a^{<}(D|I_q) \geq H_b^{<}(D|I_q)$, *if* $[x_i]^{\leq}_{\widetilde{R_a}} \subseteq [x_i]^{\leq}_{\widetilde{R_b}}$.

TABLE 4
Medical Report in Hospital 4- I_4

U	a_1	a_2	a_3	a_4	a_5	a_6
x_1	$[1.30, 1.40]$	$[0.03, 0.07]$	1.20	$[0.68, 1.10]$	$[0.43, 1.90]$	$[0.70, 0.92]$
x_2	$[1.20, 1.30]$	$[0.04, 0.11]$	$[1.20, 1.80]$	$[0.79, 1.20]$	$[0.80, 2.20]$	$[0.67, 0.88]$
x_3	$[1.10, 1.20]$	0	$[1.10, 1.20]$	$[1.20, 1.80]$	$[0.66, 0.85]$	$[0.78, 0.99]$
x_4	1.30	$[0.02, 0.09]$	$[1.10, 1.20]$	$[0.61, 0.99]$	$[1.00, 2.60]$	$[0.66, 0.88]$
x_5	$[1.30, 1.40]$	$[0.04, 0.11]$	$[1.10, 3.00]$	$[0.82, 1.20]$	$[1.40, 2.80]$	$[0.44, 0.74]$
x_6	$[1.30, 2.10]$	$[0.06, 0.16]$	$[2.00, 3.00]$	$[0.81, 1.60]$	$[0.68, 0.88]$	$[0.28, 0.61]$
x_7	$[1.20, 2.00]$	$[0.05, 0.09]$	$[1.80, 2.70]$	$[0.84, 1.50]$	$[0.78, 0.98]$	$[0.30, 0.60]$
x_8	$[1.60, 2.30]$	$[0.10, 0.20]$	$[2.20, 3.10]$	$[1.00, 1.60]$	$[0.65, 0.85]$	0
x_9	$[1.20, 1.50]$	$[0.12, 0.22]$	$[2.20, 2.70]$	$[0.90, 1.50]$	$[0.71, 0.91]$	$[0.78, 0.88]$
x_{10}	$[1.40, 2.10]$	$[0.08, 0.18]$	$[1.80, 2.70]$	0	$[1.10, 2.50]$	$[0.40, 0.70]$

4) $H_a^{>}(D|I_q) \le H_b^{>}(D|I_q)$ *or* $H_a^{>}(D|I_q) \ge H_b^{>}(D|I_q)$, *if* $[x_i]^{\gtrsim}_{R_a} \subseteq [x_i]^{\gtrsim}_{R_b}$.

Proof. Take the fuzzy dominated conditional entropy as an example. (1) If $\forall\, x_i \in U$, there exists $\frac{|[x_i]^{\lesssim}_{R_a} \cap [x_i]^{<}_{R_D}|}{|[x_i]^{\lesssim}_{R_a}|} = 1$, then we can get $H_a^{<}(D|I_q) = 0$.

(2) If $\exists\ x_i \in U$, $|[x_i]^{\lesssim}_{R_a} \cap [x_i]^{<}_{D}| = 0$, then we can get $H_a^{>}(D|I_q) = \infty$. So we have $H_a^{<}(D|I_q) < \infty$.

(3) Given $f(x,y) = -\log\frac{x}{x+y}$, we have $\frac{\partial f}{\partial x} = -\frac{y}{x(x+y)}$ and $\frac{\partial f}{\partial y} = \frac{1}{x+y}$. When $x, y > 0$, we can get $\frac{\partial f}{\partial x} < 0$ and $\frac{\partial f}{\partial y} > 0$. And we can get $\forall\ x_i \in U$, $[x_i]^{\lesssim}_{R_a} = \left([x_i]^{\lesssim}_{R_a} \cap [x_i]^{<}_{R_D}\right) \cup \left([x_i]^{\lesssim}_{R_a} \cap ([x_i]^{<}_{R_D})^c\right)$, so $\left|[x_i]^{\lesssim}_{R_a}\right| = \left|[x_i]^{\lesssim}_{R_a} \cap [x_i]^{<}_{R_D}\right| + \left|[x_i]^{\lesssim}_{R_a} \cap ([x_i]^{<}_{R_D})^c\right|$. We can get $-\log\frac{|[x_i]^{\lesssim}_{R_a} \cap [x_i]^{<}_{R_D}|}{|[x_i]^{\lesssim}_{R_a}|} = -\log\frac{|[x_i]^{\lesssim}_{R_a} \cap [x_i]^{<}_{R_D}|}{|[x_i]^{\lesssim}_{R_a} \cap [x_i]^{<}_{R_D}| + |[x_i]^{\lesssim}_{R_a} \cap ([x_i]^{<}_{R_D})^c|}$. So if we have $[x_i]^{\lesssim}_{R_a} \subseteq [x_i]^{\lesssim}_{R_b}$, then $\left|[x_i]^{\lesssim}_{R_a} \cap [x_i]^{<}_{R_D}\right| \le \left|[x_i]^{\lesssim}_{R_b} \cap [x_i]^{<}_{R_D}\right|$ and $\left|[x_i]^{\lesssim}_{R_a} \cap ([x_i]^{<}_{R_D})^c\right| \le \left|[x_i]^{\lesssim}_{R_b} \cap ([x_i]^{<}_{R_D})^c\right|$. Thus, $H_a^{<}(D|I_q) \le H_b^{<}(D|I_q)$ or $H_a^{<}(D|I_q) \ge H_b^{<}(D|I_q)$ because $\frac{\partial f}{\partial x} < 0$ and $\frac{\partial f}{\partial y} > 0$ when $x, y > 0$.

(4) The process is similar to (3). □

For simplicity and without loss of generality, in this paper we utilize the fuzzy dominated conditional entropy $H_a^{<}(D|I_q)$ to select the most important source. Thus, we have the following Definition 3 which can be used to fuse Ms-IVODS.

Algorithm 1. The Static Fusion Algorithm of Ms-IVODS

Input: $Ms - IVODS = \{(U, AT_i, V\_AT_i,\ f_i, D, V\_D,\ f\_D), i = 1, 2, \ldots, n\}$;
Output: A new fusion table.

```
for q = 1 : n do
    for each a ∈ AT do
        H_a^<(D|I_q) ← 0;
        for i=1:|U| do
            compute [x_i]^≲_{R_a} and [x_i]^<_{R_D}
            H_a^<(D|I_q) ← H_a^<(D|I_q) − (1/|U|) log ( |[x_i]^≲_{R_a} ∩ [x_i]^<_{R_D}| / |[x_i]^≲_{R_a}| )
        end
    end
end
for each a ∈ AT do
    compute i_a = argmin_{k∈{1,2,...,s}} H_a^<(D|I_k)
end
return (V_{a_1}^{i_{a_1}}, V_{a_2}^{i_{a_2}}, ..., V_{a_|AT|}^{i_{a_|AT|}}).
```

Definition 3. *Given an* $Ms - IVODS = \{\,Ms - IVOIS \cup D, V\_D, f\_D\}$. *The* i_a*-th system which is the most essential for* a *can be obtained by*

TABLE 5
The Summary of Conditional Entropy

	I_1	I_2	I_3	I_4
a_1	0.491235869	0.488578995	0.486389423	**0.485582234**
a_2	0.514888718	0.514639546	0.514689363	**0.51434019**
a_3	**0.421980985**	0.422557538	0.430053476	0.422649777
a_4	0.385685899	**0.377863609**	0.386801908	0.378310679
a_5	0.616509219	0.617479013	**0.61141654**	0.61666235
a_6	0.560092555	0.562254715	**0.559440095**	0.562678842

$$i_a = \underset{q\in\{1,2,\ldots,N\}}{\arg\min}\ H_a^{<}(D|I_q). \tag{6}$$

Based on Formula (6), a new interval-valued ordered information system can be produced. In the first step, the conditional entropy of all attributes and sources are computed, i.e., $\{H_a^{<}(D|I_q)|a \in AT, q = 1, 2, \ldots, N\}$. Next, for every attribute a, we find the source I_{i_a} whose the value of conditional entropy under a is the smallest among these s sources and transfer a of I_{i_a} to compose a new system. A visualized flow sheet of proposed fusion approach is illustrated in Fig. 1. Furthermore, the static fusion Algorithm 1 is given in the following.

Example 2. (Continued from Example 1) Based on Definition 2, we can calculate the conditional entropy of the decision attribute set under each attribute on the condition that each source is known. Take the attribute a_1 of I_1 as an example. As mentioned before, let the value of k be 1. First, the fuzzy dominated classes of these ten samples under a_1 and D are computed.

$[x_1]^{\lesssim}_{R_{a_1}} = \frac{0.50}{x_1} + \frac{0.542398}{x_2} + \frac{0.584191}{x_3} + \frac{0.529964}{x_4} + \frac{0.527472}{x_5} + \frac{0.358933}{x_6} + \frac{0.386986}{x_7} + \frac{0.268941}{x_8} + \frac{0.492501}{x_9} + \frac{0.334033}{x_{10}}$, $\ldots$, $[x_{10}]^{\lesssim}_{R_{a_1}} = \frac{0.665967}{x_1} + \frac{0.702661}{x_2} + \frac{0.736916}{x_3} + \frac{0.69211}{x_4} + \frac{0.689974}{x_5} + \frac{0.527472}{x_6} + \frac{0.557248}{x_7} + \frac{0.423115}{x_8} + \frac{0.65926}{x_9} + \frac{0.50}{x_{10}}$,

$[x_1]^{<}_{R_D} = [x_5]^{<}_{R_D} = [x_7]^{<}_{R_D} = [x_8]^{<}_{R_D} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \frac{1}{x_6} + \frac{1}{x_7} + \frac{1}{x_8} + \frac{1}{x_9} + \frac{1}{x_{10}}$, $[x_2]^{<}_{R_D} = [x_4]^{<}_{R_D} = [x_{10}]^{<}_{R_D} = \frac{0}{x_1} + \frac{1}{x_2} + \frac{0}{x_3} + \frac{1}{x_4} + \frac{0}{x_5} + \frac{0}{x_6} + \frac{0}{x_7} + \frac{0}{x_8} + \frac{0}{x_9} + \frac{1}{x_{10}}$, $[x_3]^{<}_{R_D} = [x_6]^{<}_{R_D} = [x_9]^{<}_{R_D} = \frac{0}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{0}{x_5} + \frac{1}{x_6} + \frac{0}{x_7} + \frac{0}{x_8} + \frac{1}{x_9} + \frac{1}{x_{10}}$.

Then we can get $[x_1]^{\lesssim}_{R_{a_1}} \cap [x_1]^{<}_{R_D} = \frac{0.50}{x_1} + \frac{0.542398}{x_2} + \frac{0.584191}{x_3} + \frac{0.529964}{x_4} + \frac{0.527472}{x_5} + \frac{0.358933}{x_6} + \frac{0.386986}{x_7} + \frac{0.268941}{x_8} + \frac{0.492501}{x_9} + \frac{0.334033}{x_{10}}$, $\ldots$, $[x_{10}]^{\lesssim}_{R_{a_1}} \cap [x_{10}]^{<}_{R_D} = \frac{0}{x_1} + \frac{0.702661}{x_2} + \frac{0}{x_3} + \frac{0.69211}{x_4} + \frac{0}{x_5} + \frac{0}{x_6} + \frac{0}{x_7} + \frac{0}{x_8} + \frac{0}{x_9} + \frac{0.50}{x_{10}}$.

Finally, we can compute $H_{a_1}^{<}(D|I_1) = 0.49124$ based on Definition 2.

In the same way, we respectively compute the conditional entropy of all sources for entire attributes, which are shown in Table 5. Then based on Definition 3, we can get that for the attribute a_1 and a_2, the fourth hospital I_4 is the most important source, and likewise, for a_3 is I_1; for a_4 is I_2; for a_5 and a_6 is I_3. Thus, we integrate corresponding attribute values belonging to the vitalist system into a new table as the final medical result, which is shown in Table 6.

TABLE 6
The Result of Fusion Based on Information Entropy

U	a_1	a_2	a_3	a_4	a_5	a_6
x_1	$[1.30, 1.40]$	$[0.03, 0.07]$	$[1.10, 1.20]$	$[0.67, 1.10]$	$[0.39, 1.70]$	$[0.63, 0.83]$
x_2	$[1.20, 1.30]$	$[0.04, 0.11]$	$[1.20, 1.60]$	$[0.78, 1.20]$	$[0.73, 2.00]$	$[0.61, 0.80]$
x_3	$[1.10, 1.20]$	0	$[1.00, 1.10]$	$[1.20, 1.80]$	$[0.60, 0.77]$	$[0.71, 0.90]$
x_4	1.30	$[0.02, 0.09]$	$[1.00, 1.10]$	$[0.60, 0.97]$	$[0.90, 2.40]$	$[0.60, 0.80]$
x_5	$[1.30, 1.40]$	$[0.04, 0.11]$	$[1.00, 2.80]$	$[0.80, 1.20]$	$[1.30, 2.60]$	$[0.40, 0.67]$
x_6	$[1.30, 2.10]$	$[0.06, 0.16]$	$[1.90, 2.80]$	$[0.79, 1.60]$	$[0.62, 0.80]$	$[0.25, 0.55]$
x_7	$[1.20, 2.00]$	$[0.05, 0.09]$	$[1.60, 2.50]$	$[0.83, 1.50]$	$[0.71, 0.89]$	$[0.27, 0.54]$
x_8	$[1.60, 2.30]$	$[0.10, 0.20]$	$[2.10, 2.90]$	$[1.00, 1.60]$	$[0.59, 0.77]$	0
x_9	$[1.20, 1.50]$	$[0.12, 0.22]$	$[2.10, 2.50]$	$[0.88, 1.50]$	$[0.64, 0.82]$	$[0.71, 0.80]$
x_{10}	$[1.40, 2.10]$	$[0.08, 0.18]$	$[1.60, 2.50]$	0	$[0.99, 2.30]$	$[0.36, 0.63]$

4 Dynamic Information Fusion Mechanism With the Change of Information Sources and Attributes in Ms-IVODS

For Ms-IVODS, the simultaneous variation of attributes and sources can be divided into four types. On basis of the static method, in this section, we design four dynamic mechanisms and their corresponding algorithms, which are intuitively presented in Fig. 2, Fig. 4 and Algorithms 2, 3, 4, and 5.

Case 1: Addition of information sources and deletion of conditional attributes

An incremental updating approach of Ms-IVODS is investigated in the context of adding sources and deleting attributes meanwhile. Suppose $Ms-IVODS^t$ be the Ms-IVODS at time t, $\{I_{N+1}, I_{N+2}, \ldots, I_{N+\Delta N}\}$ be the inserted sources set, and $\{a_{n+1}, \ldots, a_{n+\Delta n}\}$ be the deleted attributes set at time $t+1$. Then we have the following proposition.

Proposition 1. *For $\{a_1, a_2, \ldots, a_n\}$, the following properties are true:*

1) *If $\min_{q\in\{N+1,\ldots,N+\Delta N\}} H_a^{<}(D|I_q) \geq \min_{q\in\{1,\ldots,N\}} H_a^{<}(D|I_q)$, then $V_{a_i}^{t+1} = V_{a_i}^t$;*
2) *If $\min_{q\in\{N+1,\ldots,N+\Delta N\}} H_a^{<}(D|I_q) < \min_{q\in\{1,\ldots,N\}} H_a^{<}(D|I_q)$, then $V_{a_i}^{t+1} = V_{I_q}(a_i)$, where $q = \arg\min_{k\in\{N+1,\ldots,N+\Delta N\}} H_a^{<}(D|I_k)$ and $V_{I_q}(a_i)$ denotes the the value of a_i under I_q.*

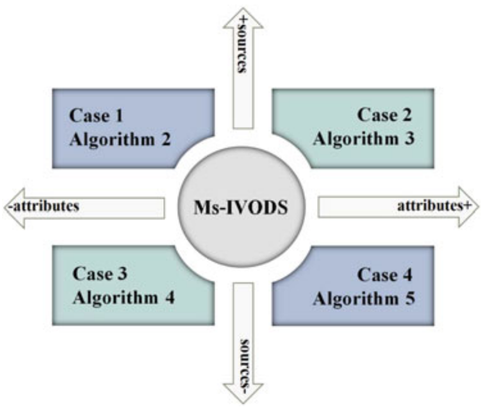

Fig. 4. The updating processes of the incremental methods.

Proof.

1) If

$$\min_{q\in\{N+1,\ldots,N+\Delta N\}} H_a^{<}(D|I_q) \geq \min_{q\in\{1,\ldots,N\}} H_a^{<}(D|I_q),$$

then we can get

$$\min_{q\in\{1,\ldots,N+\Delta N\}} H_a^{<}(D|I_q) = \min_{q\in\{1,\ldots,N\}} H_a^{<}(D|I_q),$$

so the information source which is the most important for a is unchanged. Thus, we have $V_{a_i}^{t+1} = V_{a_i}^t$.

2) If

$$\min_{q\in\{N+1,\ldots,N+\Delta N\}} H_a^{<}(D|I_q) < \min_{q\in\{1,\ldots,N\}} H_a^{<}(D|I_q),$$

then we can get that after $\{I_{N+1}, \ldots, I_{N+\Delta N}\}$ are inserted,

$$\min_{q\in\{1,\ldots,N+\Delta N\}} H_a^{<}(D|I_q) = \min_{q\in\{N+1,\ldots,N+\Delta N\}} H_a^{<}(D|I_q),$$

so the information source which is the most important for a turns to I_q, where $q = \arg\min_{k\in\{N+1,\ldots,N+\Delta N\}} H_a^{<}(D|I_k)$. □

Example 3. (Continued from Example 2) Suppose that data from the latter two hospitals are supplementary and patients just carry out the first four medical examinations in there. That means in time t, the $Ms-IVODS^t$ has two sources I_1 and I_2, where the attributes are a_1-a_6. The conditional entropy of all sources for each attribute are shown in Table 7. Then in time $t+1$, two sources I_3 and I_4 are inserted into $Ms-IVODS^t$, and in the mean time, a_5 and a_6 are removed from the original attributes set. We

TABLE 7
The Conditional Entropy at Time t in Case 1

	IS_1	IS_2
a_1	0.491235869	**0.488578995**
a_2	0.514888718	**0.514639546**
a_3	**0.421980985**	0.422557538
a_4	0.385685899	**0.377863609**
a_5	**0.616509219**	0.617479013
a_6	**0.560092555**	0.562254715

TABLE 8
The Conditional Entropy at Time $t+1$ in Case 1

	IS_1	IS_2	IS_3	IS_4
a_1	0.491235869	0.488578995	0.486389423	**0.485582234**
a_2	0.514888718	0.514639546	0.514689363	**0.51434019**
a_3	**0.421980985**	0.422557538	0.430053476	0.422649777
a_4	0.385685899	**0.377863609**	0.386801908	0.378310679

compute the conditional entropy of all sources for each attribute at time $t+1$, which are shown in Table 8.

We can find that based on Proposition 1, we only need to compute $H^{<}_{a_i}(D|I_j)$, where $i=1,2,3,4$ and $j=3,4$. For $i=1,2,3,4$, the $\min_{j\in\{1,2\}} H^{<}_{a_i}(D|I_j)$ are the former information which do not need to recalculate it.

Then we propose an incremental algorithm grounded on increasing sources and decreasing attributes in Algorithm 2.

Algorithm 2. The Incremental Algorithm With the Addition of Sources and Deletion of Attributes

Input: Original fused results $\left(V^t_{a_1}, V^t_{a_2}, \ldots, V^t_{a_n}, V^t_{a_{n+1}}, \ldots, V^t_{a_{n+\Delta n}}\right)$; Fuzzy dominated conditional entropy set $\{H^{<}_{a_i}(D|I_j), i=1,2,\ldots,n+\Delta n, j=1,2,\ldots,N\}$; Deleted attribute set $\{a_{n+1}, a_{n+2}, \ldots, a_{n+\Delta n}\}$; Inserted source set $\{I_{N+1}, I_{N+2}, \ldots, I_{N+\Delta N}\}$;
Output: An updated fusion table.

for $q = N+1 : N+\Delta N$ **do**
  for *each* $a \in \{a_1, a_2, \ldots, a_n\}$ **do**
    compute $H^{<}_{a}(D|I_q)$
  end
end
for *each* $a \in \{a_1, a_2, \ldots, a_n\}$ **do**
  if $\min_{q\in\{N+1,N+2,\ldots,N+\Delta N\}} H^{<}_{a}(D|I_q) \geq \min_{q\in\{1,2,\ldots,N\}} H^{<}_{a}(D|I_q)$ **then**
  $V^{t+1}_a = V^t_a$
  end
  if $\min_{q\in\{N+1,N+2,\ldots,N+\Delta N\}} H^{<}_{a}(D|I_q) < \min_{q\in\{1,2,\ldots,N\}} H^{<}_{a}(D|I_q)$ **then**
  $V^{t+1}_a = V_{I_q}(a), q = \arg\min_{k\in\{N+1,N+2,\ldots,N+\Delta N\}} H^{<}_{a}(D|I_q)$
  end
end
return $\left(V^{t+1}_{a_1}, V^{t+1}_{a_2}, \ldots, V^{t+1}_{a_n}\right)$.

Case 2: Addition of information sources and addition of conditional attributes

In this case, we discuss about simultaneous increase of both sources and attributes. Suppose $Ms-IVODS^t$ be the Ms-IVODS at time t, $\{I_{N+1}, I_{N+2}, \ldots, I_{N+\Delta N}\}$ be the added sources set, and $\{a_{n+1}, \ldots, a_{n+\Delta n}\}$ be the added attributes set at time $t+1$. Then we have the following propositions.

Proposition 2. *For $\{a_1, a_2, \ldots, a_n\}$, the following proposition are true:*

1) *If $\min_{q\in\{N+1,\ldots,N+\Delta N\}} H^{<}_{a}(D|I_q) \geq \min_{q\in\{1,\ldots,N\}} H^{<}_{a}(D|I_q)$, then $V^{t+1}_{a_i} = V^t_{a_i}$;*
2) *If $\min_{q\in\{N+1,\ldots,N+\Delta N\}} H^{<}_{a}(D|I_q) < \min_{q\in\{1,\ldots,N\}} H^{<}_{a}(D|I_q)$, then $V^{t+1}_{a_i} = V_{I_q}(a_i)$, where $q = \arg\min_{k\in\{N+1,\ldots,N+\Delta N\}} H^{<}_{a}(D|I_k)$ and $V_{I_q}(a_i)$ denotes the the value of a_i under I_q.*

TABLE 9
The Conditional Entropy at Time t in Case 2

	IS_1	IS_2
a_1	0.491235869	**0.488578995**
a_2	0.514888718	**0.514639546**
a_3	**0.421980985**	0.422557538
a_4	0.385685899	**0.377863609**

Proof. Proposition 2 is similar to Proposition 1, so the proof of it is similar to Proposition 1. □

Proposition 3. *For $\{a_{n+1}, a_{n+2}, \ldots, a_{n+\Delta n}\}$, we have $V_{a_i} = V_{I_q}(a_i)$, where $q = \arg\min_{k\in\{1,\ldots,N+\Delta N\}} H^{<}_{a}(D|I_k)$ and $V_{I_q}(a_i)$ denotes the the value of a_i under I_q.*

Proof. It is easy to demonstrate based on Definition 3. □

Example 4. (Continued from Example 2) Suppose that 10 patients just make four medical tests in two hospitals. And they need to have further examinations in there and retest in other two hospitals. Namely, in time t, the $Ms-IVODS^t$ has two sources I_1 and I_2, where the attributes are a_1-a_4. The conditional entropy of all sources for each attribute are shown in Table 9.

Then in time $t+1$, two sources I_3 and I_4 are inserted into $Ms-IVODS^t$, and in the mean time, a_5 and a_6 are inserted into the original attributes set. We compute the conditional entropy of all sources for each attribute at time $t+1$, which are shown in Table 10.

Similarly, we only need to compute $H^{<}_{a_i}(D|I_j)$, where $i=1,2,3,4$ and $j=3,4$ and $H^{<}_{a_i}(D|I_j)$,where $i=5,6$ and $j=1,2,3,4$. For $i=1,2,3,4$, the $\min_{j\in\{1,2\}} H^{<}_{a_i}(D|I_j)$ are the former information which do not need to recalculate it.

And then, the correlative algorithm is put forward in Algorithm 3.

Case 3: Both deletion of information sources and conditional attributes

In Case 3, we focus on the context of deleting sources and conditional attributes meanwhile. Suppose $Ms-IVODS^t$ be the Ms-IVODS at time t, $\{I_{N+1}, I_{N+2}, \ldots, I_{N+\Delta N}\}$ be the deleted sources set, and $\{a_{n+1}, a_{n+2}, \ldots, a_{n+\Delta n}\}$ be the deleted attributes set at time $t+1$. Then we have the following Proposition 4.

Proposition 4. *For $\{a_1, a_2, \ldots, a_n\}$, the following propositions are true:*

1) *If $\min_{q\in\{N+1,\ldots,N+\Delta N\}} H^{<}_{a}(D|I_q) \geq \min_{q\in\{1,\ldots,N\}} H^{<}_{a}(D|I_q)$, then $V^{t+1}_{a_i} = V^t_{a_i}$;*

TABLE 10
The Conditional Entropy at Time $t+1$ in Case 2

	IS_1	IS_2	IS_3	IS_4
a_1	0.491235869	0.488578995	0.486389423	**0.485582234**
a_2	0.514888718	0.514639546	0.514689363	**0.51434019**
a_3	**0.421980985**	0.422557538	0.430053476	0.422649777
a_4	0.385685899	**0.377863609**	0.386801908	0.378310679
a_5	0.616509219	0.617479013	**0.61141654**	0.61666235
a_6	0.560092555	0.562254715	**0.559440095**	0.562678842

2) If $\min_{q\in\{N+1,\ldots,N+\Delta N\}} H_a^{<}(D|I_q) < \min_{q\in\{1,\ldots,N\}} H_a^{<}(D|I_q)$, then $V_{a_i}^{t+1} = V_{I_q}(a_i)$, where $q = \arg\min_{k\in\{1,\ldots,N\}} H_a^{<}(D|I_k)$ and $V_{I_q}(a_i)$ denotes the the value of a_i under I_q.

Proof.

1) If

$$\min_{q\in\{N+1,\ldots,N+\Delta N\}} H_a^{<}(D|I_q) \geq \min_{q\in\{1,\ldots,N\}} H_a^{<}(D|I_q),$$

then we can get

$$\min_{q\in\{1,\ldots,N+\Delta N\}} H_a^{<}(D|I_q) = \min_{q\in\{1,\ldots,N\}} H_a^{<}(D|I_q),$$

so the information source which is the most important for a is unchanged. Thus we have $V_{a_i}^{t+1} = V_{a_i}^{t}$.

2) If

$$\min_{q\in\{N+1,\ldots,N+\Delta N\}} H_a^{<}(D|I_q) < \min_{q\in\{1,\ldots,N\}} H_a^{<}(D|I_q),$$

then we have

$$\min_{q\in\{1,\ldots,N+\Delta N\}} H_a^{<}(D|I_q) = \min_{q\in\{N+1,\ldots,N+\Delta N\}} H_a^{<}(D|I_q).$$

Thus after $\{I_{N+1}, I_{N+2}, \ldots, I_{N+\Delta N}\}$ are removed, the information source which is the most important for a turns to I_q, where $q = \arg\min_{k\in\{1,\ldots,N\}} H_a^{<}(D|I_k)$.

Based on Proposition 4, we can fast obtain the renewal of fusion results in Case 3. □

Example 5. (Continued from Example 2) Suppose that although patients have complete medical tests, we just focus on results in the first four results from the first hospitals. That is to say in time t, the $Ms-IVODS^t$ has four sources I_1- I_4, where the attributes are a_1-a_6. The conditional entropy of all sources for each attribute are shown in Table 10.

Then at time $t+1$, I_3 and I_4 are deleted, and in the mean time, a_5 and a_6 are deleted. Table 9 shows the conditional entropy of all sources for each attribute at time $t+1$.

Based on Proposition 4, for the remained attributes, we only need to utilize the former information $\min_{q\in\{3,4\}} H_a^{<}(D|I_q)$ and $\min_{q\in\{1,2\}} H_a^{<}(D|I_q)$ to update the fusion information table without any recalculation.

Then Algorithm 4 is designed.

Case 4: Deletion of information sources and addition of conditional attributes

In a similar way, this subsection talks about the situation of deleting sources and adding attributes in the meantime. Resume $Ms-IVODS^t$ be the Ms-IVODS at time t, $\{I_{N+1}, I_{N+2}, \ldots, I_{N+\Delta N}\}$ be the deleted sources set and $\{a_{n+1}, a_{n+2}, \ldots, a_{n+\Delta n}\}$ be the added attributes set at time $t+1$. Then we have the following propositions.

Proposition 5. *For $\{a_1, a_2, \ldots, a_n\}$, the following propositions are true:*

1) If $\min_{q\in\{N+1,\ldots,N+\Delta N\}} H_a^{<}(D|I_q) \geq \min_{q\in\{1,\ldots,N\}} H_a^{<}(D|I_q)$, then $V_{a_i}^{t+1} = V_{a_i}^{t}$;

2) If $\min_{q\in\{N+1,\ldots,N+\Delta N\}} H_a^{<}(D|I_q) < \min_{q\in\{1,\ldots,N\}} H_a^{<}(D|I_q)$, then $V_{a_i}^{t+1} = V_{I_q}(a_i)$, where $q = \arg\min_{k\in\{1,\ldots,N\}} H_a^{<}(D|I_k)$ and $V_{I_q}(a_i)$ denotes the the value of a_i under I_q.

Algorithm 3. The Incremental Algorithm With the Addition of Sources and Addition of Attributes

Input: Original fused results $(V_{a_1}^t, V_{a_2}^t, \ldots, V_{a_n}^t)$; Fuzzy dominated conditional entropy set $\{H_{a_i}^{<}(D|I_j), i = 1,2,\ldots,n, j = 1,2,\ldots,N\}$; Inserted source set $\{I_{N+1}, I_{N+2}, \ldots, I_{N+\Delta N}\}$; Inserted attribute set$\{a_{n+1}, a_{n+2}, \ldots, a_{n+\Delta n}\}$;
Output: An updated fusion table.

```
for q = N+1 : N + ΔN do
    for each a ∈ {a_1, a_2, ..., a_n, .., a_{n+Δn}} do
        compute H_a^<(D|I_q)
    end
end
for q = 1 : N do
    for each a ∈ {a_{n+1}, a_{n+2}, ..., a_{n+Δn}} do
        compute H_a^<(D|I_q)
    end
end
    for each a ∈ {a_1, a_2, ..., a_n} do
        if min_{q∈{N+1,N+2,...,N+ΔN}} H_a^<(D|I_q) ≥ min_{q∈{1,2,...,N}} H_a^<(D|I_q) then
            V_a^{t+1} = V_a^t
    end
        if min_{q∈{N+1,N+2,...,N+ΔN}} H_a^<(D|I_q) < min_{q∈{1,2,...,N}} H_a^<(D|I_q) then
            V_a^{t+1} = V_{I_q}(a), q = arg min_{k{N+1,N+2,...,N+ΔN}} H_a^<(D|I_q)
    end
end
for each a ∈ {a_{n+1}, a_{n+2}, ..., a_{n+Δn}} do
    compute i_a = arg min_{k∈{1,2,...,N+ΔN}} H_a^<(D|I_q)
    Let V_a = V_{I_{i_a}}(a)
end
return (V_{a_1}^{t+1}, V_{a_2}^{t+1}, ..., V_{a_n}^{t+1}, V_{a_{n+1}}, ..., V_{a+Δn}).
```

Algorithm 4. The Incremental Algorithm With the Deletion of Sources and Attributes

Input: Original fused results $(V_{a_1}^{t+1}, V_{a_2}^{t+1}, \ldots, V_{a_n}^{t+1}, V_{a_{n+1}}^{t+1}, \ldots, V_{a_{n+\Delta n}}^{t+1})$; Fuzzy dominated conditional entropy set $\{H_{a_i}^{<}(D|I_j)$,i = 1,...,n+Δn,j = 1,...,N+$\Delta N\}$; Deleted source set $\{I_{N+1}, I_{N+2}, \ldots, I_{N+\Delta N}\}$; Deleted attribute set $\{a_{n+1}, a_{n+2}, \ldots, a_{n+\Delta n}\}$;
Output: An updated fusion table.

```
for each a ∈ {a_1, a_2, ..., a_n} do
    if min_{q∈{N+1,N+2,...,N+ΔN}} H_a^<(D|I_q) ≥ min_{q∈{1,2,...,N}} H_a^<(D|I_q) then
        V_a^{t+1} = V_a^t
    end
    if min_{q∈{N+1,N+2,...,N+ΔN}} H_a^<(D|I_q) < min_{q∈{1,2,...,N}} H_a^<(D|I_q) then
        V_a^{t+1} = V_{I_q}(a), q = arg min_{k{1,2,...,N}} H_a^<(D|I_q)
    end
end
return (V_{a_1}^{t+1}, V_{a_2}^{t+1}, ..., V_{a_n}^{t+1}).
```

TABLE 11
The Comparison of the Time Complexity Between the Static and Dynamic Algorithm

Cases	Static algorithm	Dynamic algorithm
1	$O((N+\Delta N)\times n\times \|U\|^2+n)$	$O(\Delta N\times n\times \|U\|^2+n)$
2	$O((N+\Delta N)\times (n+\Delta n)\times \|U\|^2+n+\Delta n)$	$O((\Delta N\times (n+\Delta n)+N\times \Delta n)\times \|U\|^2+n+\Delta n)$
3	$O(N\times n\times \|U\|^2+n)$	$O(n)$
4	$O(N\times (n+\Delta n)\times \|U\|^2+n+\Delta n)$	$O(N\times \Delta n\times \|U\|^2+n+\Delta n)$

Proof. Proposition 5 is similar to Proposition 4, so the proof of it is similar to Proposition 4. □

Algorithm 5. The Incremental Algorithm With the Deletion of Sources and Addition of Attributes

Input: Original fused results $\left(V_{a_1}^t, V_{a_2}^t, \ldots, V_{a_n}^t\right)$; Fuzzy dominated conditional entropy set $\{H_{a_i}^{<}(D|I_j), i=1,\ldots,n, j=1,\ldots,N+\Delta N\}$; Deleted source set $\{I_{N+1}, I_{N+2}, \ldots, I_{N+\Delta N}\}$; Inserted attribute set $\{a_{n+1}, a_{n+2}, \ldots, a_{n+\Delta n}\}$;
Output: An updated fusion table.

for $q = 1:N$ **do**
  for *each* $a \in \{a_{n+1}, a_{n+2}, \ldots, a_{n+\Delta n}\}$ **do**
    compute $H_a^{<}(D|I_q)$
  end
end
for *each* $a \in \{a_1, a_2, \ldots, a_n\}$ **do**
  if $\min_{q\in\{N+1,N+2,\ldots,N+\Delta N\}} H_a^{<}(D|I_q) \geq \min_{q\in\{1,2,\ldots,N\}} H_a^{<}(D|I_q)$ **then**
    $V_a^{t+1} = V_a^t$
  end
  if $\min_{q\in\{N+1,N+2,\ldots,N+\Delta N\}} H_a^{<}(D|I_q) < \min_{q\in\{1,2,\ldots,N\}} H_a^{<}(D|I_q)$ **then**
    $V_a^{t+1} = V_{I_q}(a), q = \arg\min_{k\{1,2,\ldots,N\}} H_a^{<}(D|I_q)$
  end
end
for *each* $a \in \{a_{n+1}, a_{n+2}, \ldots, a_{n+\Delta n}\}$ **do**
    compute $i_a = \arg\min_{k\in\{1,2,\ldots,N\}} H_a^{<}(D|I_q)$
    Let $V_a = V_{I_{i_a}}(a)$
end
return $\left(V_{a_1}^{t+1}, V_{a_2}^{t+1}, \ldots, V_{a_n}^{t+1}, V_{a_{n+1}}, \ldots, V_{a+\Delta n}\right)$.

Proposition 6. *For $\{a_{n+1}, a_{n+2}, \ldots, a_{n+\Delta n}\}$, we have $V_{a_i} = V_{I_q}(a_i)$, where $q = \arg\min_{k\in\{1,2,\ldots,N\}} H_a^{<}(D|I_k)$ and $V_{I_q}(a_i)$ denotes the the value of a_i under I_q.*

Proof. It is easy to confirm based on Definition 3. □

Example 6. (Continued from Example 2) Suppose that 10 patients have already made four examinations in four hospitals but we only pay attention to the data from the first two hospitals and their corresponding six indices. That means in time t, the $Ms-IVODS^t$ has four sources I_1- I_4, where the attributes are a_1-a_4. The conditional entropy of all sources for each attribute are shown in Table 8.

Then in time $t+1$, two sources I_3 and I_4 are removed and in the mean time, a_5 and a_6 are inserted. We compute the conditional entropy of all sources for each attribute at time $t+1$, which are shown in Table 7.

Based on Propositions 5 and 6, we only need to compute $H_{a_i}^{<}(D|I_j)$, where $i=5,6$ and $j=1,2$. For $i=1,2,3,4$, the $\min_{j\in\{1,2\}} H_{a_i}^{<}(D|I_j)$ and $\min_{j\in\{3,4\}} H_{a_i}^{<}(D|I_j)$ are the former information which do not need to recalculate it.

And on the basis of the above, we come up with relevant mechanism Algorithm 5.

Further, the time complexity comparison are given respectively under the dynamic conditions and the static one in Table 11.

5 Experimental Analysis

In this part, we evaluate the Ms-IVODS static fusion technique and relative incremental methods. Twelve datasets are downloaded from UCI, which are shown in Table 12. And all experiments are conducted on a private computer with an i5-11300H processor, 16GB, and a Windows 11 operating system, using a Python 3.7 platform. Before starting the experiments, we employ the approach in [45] to produce multi-source interval-valued information systems. Then we generate 20 sources for each dataset. And it should

TABLE 12
The Description of Datasets

No.	Data sets	Abbreviation	Samples	Attributes	Classes
1	Ionosphere	Iono	351	34	2
2	Auto MPG	AM	398	8	3
3	Hill-Valley	HV	606	101	2
4	Credit Approval	CA	690	16	2
5	Wine Quality-red	WQR	1599	12	6
6	Car Evaluation	CE	1728	7	4
7	Cardiotocography	Card	2126	22	10
8	Abalone	Abalone	4177	9	3
9	Wine Quality-white	WQW	4898	12	7
10	Page Blocks Classification	PB	5473	10	5
11	Shill Bidding	Bid	6321	13	2
12	Nursery	Nursery	12960	9	5

TABLE 13
The Classification Accuracy of the Fusion Results of the Four Fusion Methods

Datasets	KNN			
	MeanF	MaxF	MinF	CeF
Iono(k=15)	57.5±9.2	57.8±9.8	58.4±7.8	**59.2±8.0**
AM(k=6)	60.3±8.0	58.3±7.8	59.1±7.9	**62.3±8.5**
HV(k=17)	46.5±6.1	46.2±6.6	46.2±5.2	**47.7±4.7**
CA(k=17)	65.4±5.4	65.5±5.9	65.9±5.5	**68.4±6.1**
WQR(k=11)	41.5±3.3	42.0±4.0	41.3±3.8	**43.1±3.8**
CE(k=23)	69.5±3.9	69.6±3.6	69.7±3.8	**69.9±3.9**
Card(k=4)	35.6±1.3	34.9±2.0	36.8±1.8	**37.9±2.3**
Abalone(k=24)	34.6±1.1	36.2±2.1	35.0±1.4	**36.3±1.5**
WQW(k=43)	42.9±1.6	43.0±1.5	42.8±2.1	**43.8±2.4**
PB(k=2)	88.9±1.3	88.9±1.5	89.0±1.2	**89.1±1.0**
Bid(k=1)	80.9±1.8	80.8±1.7	80.9±1.8	**81.0±2.1**
Nursery(k=3)	52.3±1.6	**53.2±1.7**	50.9±1.4	52.4±1.2

TABLE 14
The Classification Accuracy of the Fusion Results of the Four Fusion Methods

Datasets	PNN			
	MeanF	MaxF	MinF	CeF
Iono(σ= 0.1)	57.8±6.5	57.8±8.4	59.5±5.8	**59.8±7.9**
AM(σ= 0.05)	47.2±8.9	47.7±8.0	**51.7±8.3**	51.3±6.5
HV(σ= 0.8)	52.5±3.7	52.3±3.3	**53.5±3.3**	53.3±3.4
CA(σ= 0.6)	60.6±6.4	57.8±4.9	64.2±7.6	**65.1±6.5**
WQR(σ= 0.1)	41.6±3.6	42.0±4.5	40.4±4.3	**42.3±4.4**
CE(σ= 0.2)	69.7±4.0	69.7±3.8	69.8±3.8	**70.0±3.9**
Card(σ=0.1)	39.8±3.2	38.3±3.4	41.1±3.4	**42.4±3.2**
Abalone(σ= 0.05)	33.1±1.2	33.7±1.9	34.2±1.7	**34.7±2.0**
WQW(σ= 0.25)	44.8±2.0	44.9±2.1	44.2±1.8	**45.0±2.0**
PB(σ= 0.01)	82.0±1.5	82.4±1.5	82.1±1.3	**82.4±1.2**
Bid(σ= 0.2)	89.2±1.3	**89.2±1.3**	89.0±1.4	89.1±1.3
Nursery(σ= 0.5)	51.4±1.0	50.5±0.9	52.5±1.3	**54.5±1.5**

be noted that for numerical attributes, the order relation is natural. While for categorical attributes, the preference relation is defined based on the semantics of the attributes.

5.1 The Analysis of Fusion Effectiveness

This subsection aims to verify the effectiveness of the proposed fusion method, which is denoted as CeF. And the verification is composed of two respects. For one thing, we compare CeF with three common fusion methods, including MeanF, MaxF and MinF via classification accuracy and corresponding P-values. And the three mechanisms are given below:

1) $MaxF_a(x) = [\min_{i\in\{1,2,\ldots N\}} f_i^L(x,a), \max_{i\in\{1,2,\ldots N\}} f_i^U(x,a)]$,
2) $MinF_a(x) = [\max_{i\in\{1,2,\ldots N\}} f_i^L(x,a), \min_{i\in\{1,2,\ldots N\}} f_i^U(x,a)]$,
3) $MeanF_a(x) = [\frac{1}{N}\sum_{i=1}^N f_i^L(x,a), \frac{1}{N}\sum_{i=1}^N f_i^U(x,a)]$.

For another thing, we evaluate whether the performance of CeF on effectiveness has an advantage over HF, the state-of-art fusion method of multi-source interval-valued data designed in [45]. In addition to the above accuracy comparison, AP and AQ, as the indicators of precision and quality of approximation classification, are used to assess the benefits of CeF as well. And it is common knowledge that many existing classifiers cannot directly cope with interval-valued data. So in [50], the extended KNN and PNN classifiers are introduced to address interval-valued data. We can adjust the parameter k of the KNN classifier and σ of the PNN classifier to obtain the optimal classification result. The mean and standard deviation of classification accuracy are reported using a ten-fold cross-validation method. And the Wilcoxon signed-rank test is used to assess if the proposed approach outperforms HF statistically. Then corresponding P-values are got.

First, we give an overview of the comparison outcome of CeF and three common methods: MeanF, MaxF and MinF. The results are shown in Tables 13 and 14. We can discover that in the KNN classifier, the proposed fusion method outperforms the other three common fusion methods in eleven datasets. In dataset Nursery, although slightly fewer than MaxF, CeF is still better than MeanF and MinF. In the PNN classifier, our method performs better than the others in nine datasets. For the other three datasets, the performance of CeF all ranks second. Additionally, as shown in Table 15, when setting a significance level of 10%, most of the P-values are smaller than 0.1, especially in PNN classifier. And that manifests the designed fusion mechanism CeF is statistically better than the other three common approaches in most situations.

Next, CeF and HF are compared on classification precision. Then the results are shown in Tables 16 and 17.

TABLE 15
P-Values of the Comparison Results in Classification Accuracy

Datasets	KNN			PNN		
	H_1:CeF>MeanF	H_1:CeF>MinF	H_1:CeF>MaxF	H_1:CeF>MeanF	H_1:CeF>MinF	H_1:CeF>MaxF
Iono	0.3349	0.2591	0.3357	0.5000	**0.0289**	0.1163
AM	0.1465	**0.0700**	**0.0615**	**0.0614**	0.5834	**0.0616**
HV	**0.0838**	**0.0145**	0.1867	**0.0700**	0.9772	**0.0526**
CA	**0.0103**	**0.0374**	**0.0028**	**0.0046**	0.1704	**0.0029**
WQR	**0.0765**	**0.0840**	**0.0860**	0.1201	**0.0328**	0.2491
CE	**0.0856**	0.2940	**0.0907**	**0.0445**	**0.0987**	**0.0131**
Card	**0.0083**	**0.0331**	**0.0046**	**0.0040**	**0.0122**	**0.0045**
Abalone	**0.0071**	**0.0332**	0.5203	**0.0377**	**0.0963**	0.1106
WQW	**0.0515**	**0.0416**	0.1309	**0.0482**	**0.0029**	0.2948
PB	0.2195	0.1158	0.2002	**0.0747**	0.5472	**0.0770**
Bidding	0.2766	0.2375	0.3834	0.6082	0.9693	**0.0502**
Nursery	**0.0040**	0.9839	0.5237	**0.0000**	**0.0000**	**0.0000**

TABLE 16
The Classification Accuracy of the Fusion Results of HF and CeF

Datasets	KNN		Datasets	PNN	
	HF	CeF		HF	CeF
Iono(k=20)	55.0±8.3	**59.2±7.9**	Iono(σ=0.5)	56.7±4.6	**63.3±7.3**
AM(k=6)	58.3±9.1	**62.3±8.5**	AM(σ =0.5)	52.0±9.2	**62.6±7.2**
HV(k=17)	45.5±6.1	**47.7±4.7**	HV(σ=0.8)	46.4±4.5	**53.3±3.4**
CA(k=3)	**65.8±3.1**	64.6±4.3	CA(σ= 0.1)	**63.6±2.9**	62.8±5.8
WQR(k=11)	42.3±3.0	**43.1±3.8**	WQR(σ= 0.1)	39.5±3.4	**42.3±4.4**
CE(k=23)	69.3±3.7	**69.9±3.9**	CE(σ= 0.2)	68.5±2.9	**70.0±3.9**
Card(k=4)	**39.6±2.7**	37.9±2.3	Card(σ= 0.1)	**44.7±2.7**	42.4±3.2
Abalone(k=24)	35.4±2.0	**36.3±1.5**	Abalone(σ= 0.05)	34.1±2.0	**34.7±2.0**
WQW(k=43)	43.2±1.5	**43.8±2.4**	WQW(σ= 0.25)	35.9±1.8	**45.0±2.0**
PB(k=2)	88.8±1.3	**89.1±1.0**	PB(σ=0.2)	82.4±1.9	**89.7±1.3**
Bid(k=1)	80.7±1.4	**81.0±2.2**	Bid(σ=0.2)	**89.3±1.3**	89.3±1.3
Nursery(k=1)	58.0±1.4	**58.3±1.7**	Nursery(σ=0.2)	58.4±1.2	**59.8±1.4**

Table 16 indicates that no matter in KNN or in PNN, CeF always has a better performance in most datasets. That is to say the proposed method CeF can generate a fusion table with higher precision for most datasets regardless of the classifier. However, Table 17 reveals that our method is notably better than HF in PNN classifier. While in KNN case, CeF has no obvious advantage over the state-of-art fusion approach HF on classification accuracy .

Then, to demonstrate the benefits of our methods over the state-of-the-art fusion methodology HF, we compare the two fusion methods in terms of AP and AQ. AP and AQ can be used to effectively indicate the accuracy and quality of approximation classification. The greater the AP and AQ are, the more precise and high-quality the approximation classification is. We define a novel relation R_α on U:$R_\alpha = \left\{(x, y) \in U \times U | \frac{dis(x,y)}{\max_{z \in U}\{dis(x,z)\}} \le \alpha \right\}$, where $dis(x, y)$ denotes the distance between x and y. The distance function between any two interval-valued data is proposed in [51], and the distance between any two general trapezoidal fuzzy numbers is defined in [52].

We use the relation to calculate AP and AQ, where α is defined to be between 0.05 and 0.5 with a 0.05 step. The AP and AQ of fusion results between CeF and HF are shown in Figs. 5 and 6. From the two figures, it is evident no matter what the value of α is, AP and AQ of CeF are larger than HF. That is to say, CeF is always keeping its strengths and it has obvious advantage over the state-of-the-art fusion approach HF in approximation classification.

TABLE 17
P-Values of the Comparison Results in Classification Accuracy Between CeF and HF

Datasets	KNN	Datasets	PNN
	H_1:CeF>HF		H_1:CeF>HF
Iono(k=20)	**0.0296**	Iono(σ=0.5)	**0.0111**
AM(k=6)	0.1298	AM(σ=0.5)	**0.0082**
HV(k=17)	0.1355	HV(σ=0.8)	**0.0064**
CA(k=3)	0.7032	CA(σ=0.1)	0.7793
WQR(k=11)	0.3050	WQR(σ=0.1)	**0.0703**
CE(k=23)	**0.0912**	CE(σ=0.2)	**0.0696**
Card(k=4)	0.9382	Card(σ=0.1)	0.9971
Abalone(k=24)	0.2701	Abalone(σ=0.05)	0.2703
WQW(k=43)	0.2703	WQW(σ=0.25)	**0.0030**
PB(k=2)	0.1106	PB(σ=0.02)	**0.0029**
Bid(k=1)	0.2067	Bid(σ=0.2)	0.9772
Nursery(k=1)	0.5236	Nursery(σ=0.2)	**0.0205**

5.2 The Analysis of Efficiency

In this subsection, designed to demonstrate the efficiency of incremental CeF, four modified approaches are compared with the static one in accordance with computation time and speed-up ratio. The update of dataset is simulated as follows. As for the variation of sources, when source increase, we regard the 50% data sources as the origin, and the number of sources is increased by 10% from the rest. When sources decrease, the whole dataset is considered the basic and is successively reduced by 10%. It is the same for the change of attributes, while we choose the variable ranges based on the attributes' number of each dataset.

As for computational time, the results are reported in Figs. 7, 8, 9, and 10. And it is apparent that the fusion time of incremental methods is remarkably smaller than that of the static approach. On account of the application of previous knowledge, dynamic mechanisms are able to avoid repeating calculating and dramatically save computation time.

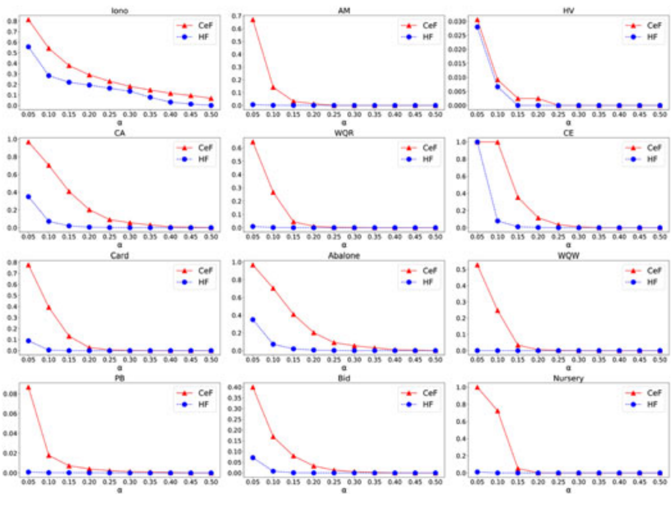

Fig. 5. AP of CeF and HF.

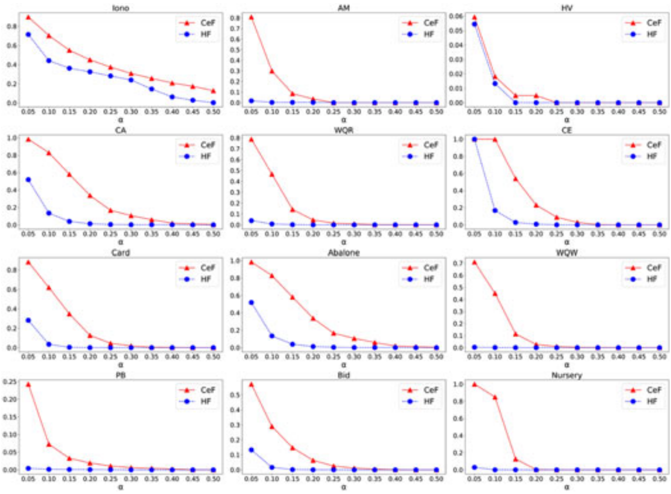

Fig. 6. AQ of CeF and HF.

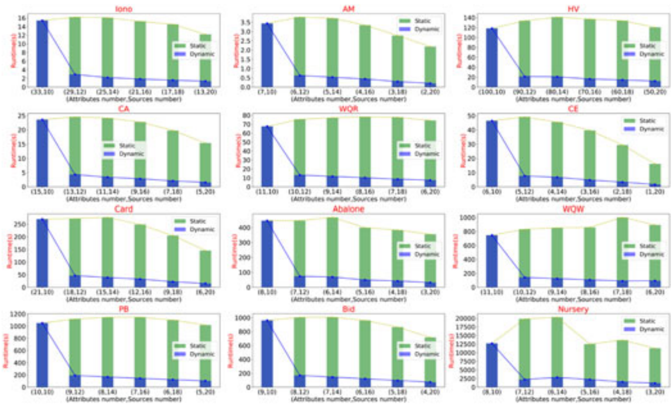

Fig. 7. Runtime time of case 1.

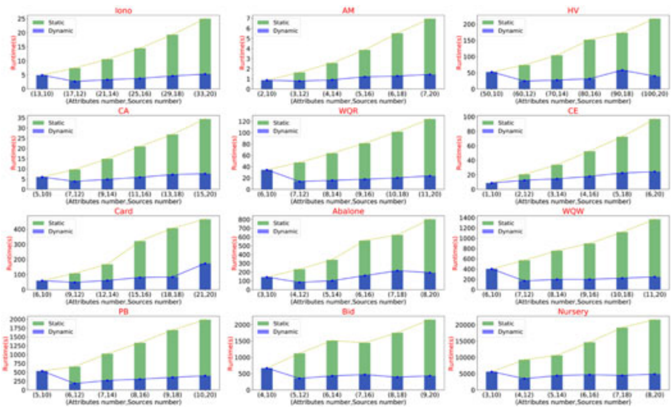

Fig. 8. Runtime time of case 2.

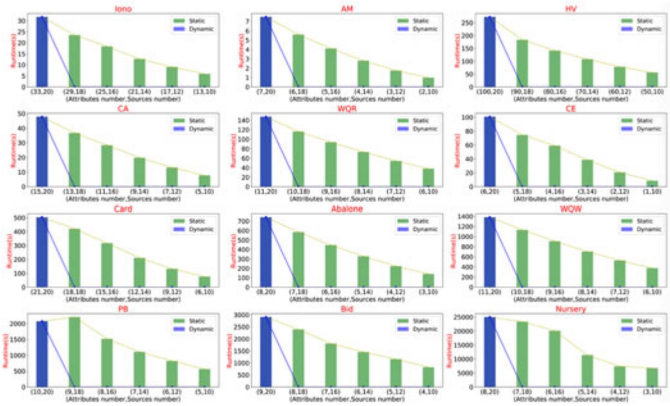

Fig. 9. Runtime time of case 3.

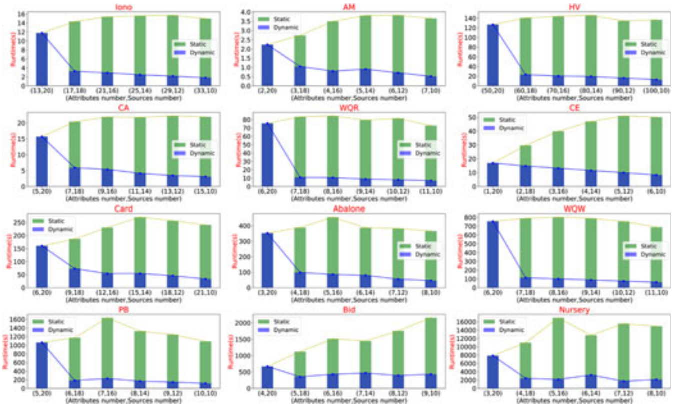

Fig. 10. Runtime time of case 4.

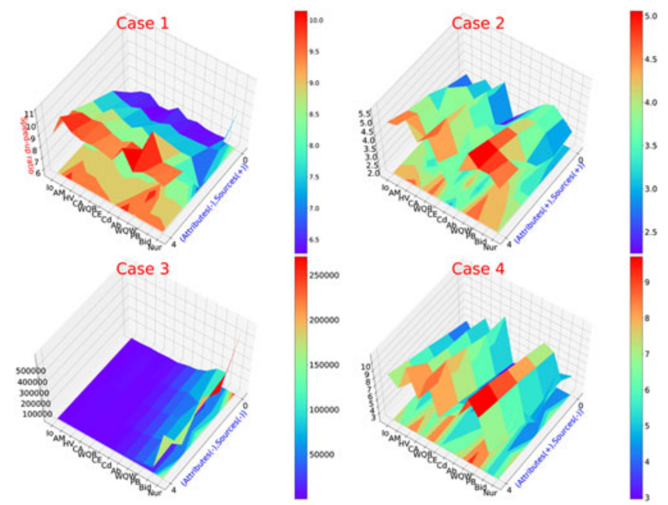

Fig. 11. The speed-up ratio of case 1-4.

Subsequently, we discuss the efficiency demonstration from another aspect of speed-up ratio and the results are displayed in Fig. 11. The x-axis represents the process through which attributes and sources vary, where $0 \rightarrow 4$ is the change direction, Attributes number(-) and Sources number(-) denote deleting attributes and sources respectively, and Attributes number(+) and Sources number(+) denote adding attributes and sources. The y-axis stands for abbreviations of datasets, and the z-axis is the reflection of the speed-up ratio. From the results, we can get the dynamic updating approach yield speedup ranging from 5.5 to 11$\times$ over the static fusion approach in Case 1. And the ratios in Case 2-4 are 1.7-5.5$\times$, 63-598729$\times$ and 2-11$\times$ respectively. It should be noted the speed-up ratio of the dynamic updating approach for Case 3 is abnormally high since the static fusion approach needs $N \times n$ calculations, where N denotes the number of remaining sources and n denotes the number of remaining attributes. However, in Case 3, the dynamic updating approach only needs to seek out the relative information rather than generate something new to update the fusion results, consequently, the improved efficiency of the dynamic updating approach for Case 3 is very high. Furthermore, the Wilcoxon signed-rank test is utilized to demonstrate whether the dynamic algorithms can significantly reduce the runtime of fusion. The P-value of all tests are $4.56 \times e^{-44}$. Given a significance level of 10%, we can find that the result is statistically significant. In general, the four dynamic fusion mechanisms can markedly reduce the runtime of fusion.

6 Conclusion and Future Work

Interval-valued ordered data from several sources are ubiquitous in real-world applications, characterizing unpredictability of real-life situations. And that is the reason why it is vital to research multi-source interval-valued ordered data fusion. Through defining a fuzzy dominance relation and relative conditional entropy for interval-valued ordered data, this study presents a conditional entropy fusion strategy for Ms-IVODS. In addition, the four incremental techniques of dynamic fusion are designed for simultaneously varying information sources and attributes. Then we compare the time complexity of four dynamic methods to the static fusion approach. Fig. 4 shows the relationships of the whole five algorithms. Finally, twelve experiments demonstrate that the proposed fusion technique beats three conventional fusion approaches in terms of classification accuracy and exceeds the state-of-the-art fusion approach in terms of approximation classification. Moreover, the efficiency study indicates that these dynamic approaches updating fusion table can efficiently reduce the fusion process running time.

According to the deficiencies presented in the Introduction, there are two aspects for our future research. For one thing, we try to develop incremental mechanisms with the simultaneous changes of sources, attributes and objects. For another thing, we will attach greater importance to the practical applications such as the assessment of the treatment technologies in health care waste management.

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