Two-Way Concept-Cognitive Learning Method: A Fuzzy-Based Progressive Learning

Weihua Xu, Doudou Guo, Yuhua Qian, and Weiping Ding

***Abstract*—Granular computing (GrC) and two-way concept learning (TCL) are influential studies of knowledge processing and cognitive learning. A central notion of two-way concept learning is learning concepts from an arbitrary information granule. Although TCL has been widely adopted for concept learning and formal concept analysis in a fuzzy context, the existing studies of TCL still have some issues: the sufficient and necessary granule concept is only obtained from the necessary granule or sufficient granule concept; and the cognitive mechanism ignores integrating past experiences into itself to deal with dynamic data. Meanwhile, concept-cognitive learning (CCL) method still faces challenges, such as incomplete cognitive and weak generalization ability. This article proposes a novel two-way CCL (TCCL) method for dynamic concept learning in a fuzzy context for these problems and challenges. Unlike TCL, fuzzy-based TCCL (F-TCCL) is more flexible and less time consuming to learn granule concepts from the given clue, and meanwhile, it is good at dynamic concept learning. Moreover, we design a fuzzy-based progressive learning mechanism within this framework under the dynamic environment. Some numerical experiments on public datasets verify the effectiveness of our proposed method. The considered framework can provide a convenient novel method for researching two-way learning and CCL.**

***Index Terms*—Concept-cognitive learning (CCL), formal concept analysis, fuzzy set, granular computing, two-way learning.**

I. Introduction

COGNITIVE science is important for researching the concept of artificial intelligence [1]. Since 1956, researchers in various fields have researched this subject from their professional perspectives, and even within the same field, there were various theoretical frameworks, e.g., information processing theory [2], three language description models [3], logic theory machine, etc. Thus, it can be said that this is a highly controversial area. During the past 66 years, we have witnessed a wide variety of theories, frameworks, models, and viewpoints of cognitive sciences. For instance, concept-cognitive learning (CCL), the idea of cognitive through concept formation and learning to reveal the systematic law of the human brain, is an effective cognitive mechanism, such as abstract concept [4], [5], Wille's concept [6], [7], rule concept [8], object-oriented concept [9], [10], AFS-concept [11], and three-way concept [12], [13].

Granular computing, a new computing paradigm in intelligent information processing, is particular in that it can imitate the cognitive mechanism of humans dealing with complex problems [14]. It focuses on the structure of multigranularity space and the calculation between granularity, which belongs to cognitive mechanism research at a higher level of human beings and has strong practicability [15]. Up to now, the outline of granular computing appears basically, including formal concept analysis theory [6], fuzzy set theory [16], rough set theory [17], etc. The basic idea of granular computing has been widely used in many disciplines and fields, especially in interdisciplinary applications. A novel concept cognitive model based on granular computing, coined by Zhang in his seminal paper [18], is a convenient tool for researching cognitive models and concept learning. Subsequently, Wang [4] provided the basic algebraic model of concept learning from the perspective of cognitive computing. Yao [9] examined the framework of concept learning from cognitive informatics and granular computing perspectives and pointed out that the systematic study of concept learning based on knowledge discovery should include three sublevels: the philosophy level, the algorithm/technique level, and the application level. Through a fruitful marriage of GrC and concept learning, it is possible to investigate the cognitive mechanism of concept learning and knowledge discovery, which becomes the rudiments of cognitive concept learning.

The basic idea of cognitive concept learning is to learn concepts from given clues by specific cognitive models and to reveal the systematic law of concept learning in the human brain. From 2013 to 2018, this line of thought fostered several cognitive concept learning models and methods. For example, Xu [19] studied a two-way learning system for information granule transformations. Li [20] proposed a granule concept learning model from a cognitive viewpoint. Shivhare and Cherukuri [21] introduced a three-way conceptual method for cognitive memory functionalities. With the in-depth research of cognitive concept learning theory in model, method, and application, more and more scholars join in the research. In particular, CCL, the

Manuscript received 12 January 2022; revised 16 August 2022; accepted 10 October 2022. Date of publication 20 October 2022; date of current version 2 June 2023. This work was supported in part by the National Natural Science Foundation of China under Grant 61976245 and Grant 61976120; in part by the Natural Science Key Foundation of Jiangsu Education Department under Grant 21KJA510004; and in part by the Chongqing Postgraduate Research and Innovation Project under Grant CYB22153. *(Corresponding author: Doudou Guo.)*

Weihua Xu and Doudou Guo are with the College of Artificial Intelligence, Southwest University, Chongqing 400715, China (e-mail: chxuwh@gmail.com; doudou876517690@126.com).

Yuhua Qian is with the Institute of Big Data Science and Industry, Shanxi University, Taiyuan 030006, China (e-mail: jinchengqyh@126.com).

Weiping Ding is with the School of Information Science and Technology, Nantong University, Nantong 226019, China (e-mail: dwp9988@163.com).

Color versions of one or more figures in this article are available at https://doi.org/10.1109/TFUZZ.2022.3216110.

Digital Object Identifier 10.1109/TFUZZ.2022.3216110

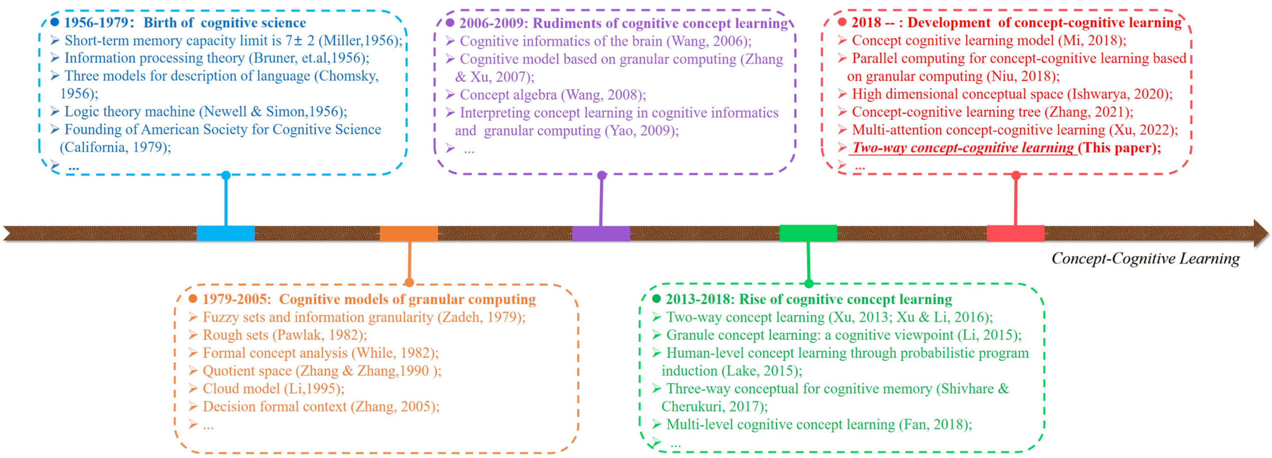

Fig. 1. Development stage of CCL: A granular computing viewpoint.

practical marriage of cognitive concept learning and machine learning, further broadens the research horizon of this field. Against this background, Mi successively proposed a series of CCL methods such as the incremental CCL model [22], concurrent CCL model [23], etc. Xu [24] built multiattention CCL in a formal context. In order to show the development stage of the CCL theory more clearly, we summarized the development stage of this theory from a perspective of granular computing, as shown in Fig. 1. Note that Fig. 1 may be viewed as a list of some examples rather than an exhaustive summary.

Note that two-way learning [19], [25], [26] is another representative study of CCL, and its advantage lies in learning more from the unknown through a pair of cognitive operators (i.e., extent-intent and intent-extent) and mapping it into granule concept space for different semantic interpretation (i.e., necessary, sufficient, sufficient, and necessary granule concept). It is worth mentioning that the two-way learning model is a vital CCL system to describe human cognitive processes. Nevertheless, two-way learning also has some limitations, mainly reflected in the following:

1) concept learning is mainly through the two-way transformation of information granules, and the transformation mechanism is too complicated;
2) the cognitive system learning a concept from a given clue, regardless of discussing whether the pair of the object and attribute sets is concept-inducible or not, let alone the learning accuracy;
3) the learning mechanism emphasizes more the static environment, and the initial clues need to be fixed, which is not suitable for dynamic and real-time updating concept learning, that is, it cannot ignores integrating past experiences into itself to deal with dynamic data.

Due to the challenges mentioned previously, the relevant study process stagnates, and only a few research findings are available [25], [26]. Of particular note is that the concept of cognitive learning research is at an early stage. Although there have been many significant achievements, it also appears very important to improve and enrich the area of CCL from various theories, frameworks, models, and viewpoints. Therefore, in this article, we again put forward the CCL system of two-way learning and point out and solve the problems existing in the current study is one of the purposes. Another motivation in this article is comparing TCCL with many other CCL model differences and its unique place.

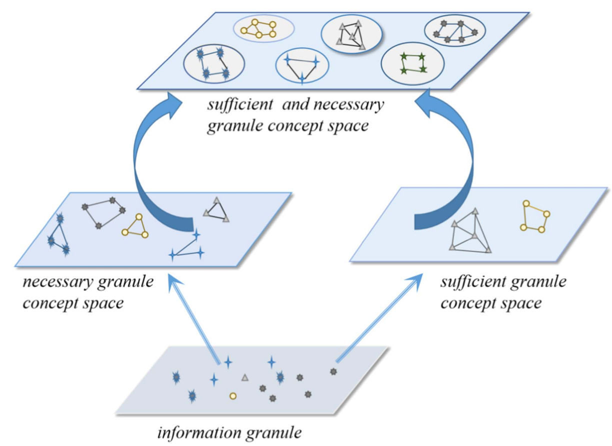

Fig. 2. Two-way concept-cognitive learning (TCCL). It consists of three levels and two stages: the first stage is learning necessary and sufficient granule that stores in the middle level from arbitrary information granule (i.e., bottom level); the second stage is learning sufficient and necessary fuzzy granule concepts through the granule stored in the necessary and sufficient concept space, that is, the middle-to-top level. Therefore, the fuzzy granule concepts can be learned according to the aforementioned clues in Fig. 1.

Following the thought of two-way learning, we build our fuzzy-based TCCL (F-TCCL) by introducing the TCL and GrC to progressive learning concepts from given clues. Furthermore, we summarize the main contributions of this article as follows.

1) Point out granulation relation based on two-way learning, no longer through the transformation between different granule concepts. Compared with TCL, F-TCCL pays more attention to concept learning through the cognitive viewpoint and cross-level learning (see Fig. 2), and the experiment also verifies that this method is effective.

2) F-TCCL emphasizes learning concepts from clues. Hence, our method for specific tasks or clues is more advantageous than other CCL methods. Moreover, this study builds the progressive learning by the granulation mechanism for TCCL that is more effective than the existing study of TCL, especially in dynamic data.
3) The fuzzy granule concept is incorporated into the cognitive mechanism of F-TCCL, and a fuzzy-based progressive learning method is built for the dynamic data. Compared with other two-way learning systems, this article proposes the progressive learning of F-TCCL that can integrate the past experience into itself by the update mechanisms to reduce the complexity of the learning system.

The remainder of this article is organized as follows. Section II briefly reviews some basic notions about two-way learning and CCL. Section III presents a novel cognitive mechanism based on two-way learning. The theory of F-TCCL for dynamic learning and its corresponding granulation relation is presented in Section IV. The experimental analysis is given in Section V. Finally, Section VI concludes this article.

II. Preliminaries

This section briefly reviews some basic notions: 1) fuzzy formal context, 2) two-way learning, and 3) granule concept learning. A detailed discussion of them can be found in [20], [25], [27], [28], and [29].

Before we start this section, it is necessary to claim that this article discusses the CCL model in a fuzzy formal context without a regular formal context, and the pseudofuzzy concept may be called a fuzzy concept when no confusion exists.

A. Fuzzy Formal Context

In this subsection, we start with the notion of a fuzzy formal context [27], [30], and the description of several essential notions is shown as follows.

Let U be a nonempty and finite set, a fuzzy set $\widetilde{X}$ of U can be defined as follows:

$$\widetilde{X} = \{< x, \mu_{\widetilde{X}}(x) > | x \in U\}$$

where $\mu_{\widetilde{X}} : U \to [0, 1]$, $\mu_{\widetilde{X}}(x)$ denotes the membership degree of the object x with respect to $\widetilde{X}$, and $\mu^c_{\widetilde{X}}(x) = 1 - \mu_{\widetilde{X}}(x)$ is the nonmembership.

A fuzzy formal context is a triple $(U, A, \widetilde{I})$, where $U = \{x_1, x_2, \ldots, x_n\}$ and $A = \{a_1, a_2, \ldots, a_m\}$ are two nonempty finite object and attribute sets, and $\widetilde{I} = \{< (x, a), \mu_{\widetilde{I}}(x, a) > | (x, a) \in U \times A\}$ is a fuzzy binary relation. For any $\widetilde{I}(x, a) \in U \times A$ has a membership degree $\mu_{\widetilde{I}}(x, a) \in [0, 1]$. We denote $\widetilde{I}(x, a) = \mu_{\widetilde{I}}(x, a)$ for convenience. Similarly, the complement of $\widetilde{I}$ can be denoted by $\widetilde{I} = \{< (x, a), 1 - \mu_{\widetilde{I}}(x, a) > | (x, a) \in U \times A\}$. Given $\widetilde{I}(x, a)$ and $\widetilde{I}(y, a)$, the $\widetilde{I}(x, a) \geqslant \widetilde{I}(y, a) \Leftrightarrow \mu_{\widetilde{I}}(x, a) \geqslant \mu_{\widetilde{I}}(y, a)$. As a generalization of the classical formal context, fuzzy formal context naturally has various operations and properties (see [25], [27], [28], and [29]).

A fuzzy formal context $(U, A, \widetilde{I})$. For any $X \subseteq U$ and $\widetilde{B} \in \Gamma^A$, the derivation operator $(\cdot)^*$ can be defined as follows:

$$X^*(a) = \bigwedge_{x \in X} \widetilde{I}(x, a), a \in A$$

$$\widetilde{B}^* = \{x \in U | \forall a \in A, \widetilde{B}(a) \leqslant \widetilde{I}(x, a)\}$$

where Γ^A is the union of all fuzzy sets in A.

Therefore, $(U, A, \widetilde{I})$ is a fuzzy formal context, a pair $(X, \widetilde{B})$ is called a fuzzy formal concept or fuzzy concept, if only if $X^* = \widetilde{B}$ and $\widetilde{B}^* = X$, where X is the extent and $\widetilde{B}$ is the intent of the concept $(X, \widetilde{B})$. Obviously (X^{**}, X^*) and $(\widetilde{B}^*, \widetilde{B}^{**})$ are fuzzy concepts. The fuzzy concept lattice $\widetilde{L}(U, A, \widetilde{I})$ is the union of all fuzzy concept in $(U, A, \widetilde{I})$. For any fuzzy concept $(X_1, \widetilde{B}_1), (X_2, \widetilde{B}_2) \in \widetilde{L}(U, A, \widetilde{I})$, the ordered by $(X_1, \widetilde{B}_1) \leqslant (X_2, \widetilde{B}_2) \Leftrightarrow X_1 \subseteq X_2 \Leftrightarrow \widetilde{B}_1 \subseteq \widetilde{B}_2$. Moreover, if the meet and join are given by

$$(X_1, \widetilde{B}_1) \wedge (X_2, \widetilde{B}_2) = (X_1 \cap X_2, (\widetilde{B}_1 \cup \widetilde{B}_2)^{**})$$

$$(X_1, \widetilde{B}_1) \vee (X_2, \widetilde{B}_2) = ((X_1 \cup X_2)^{**}, \widetilde{B}_1 \cap \widetilde{B}_2)$$

then the fuzzy concept lattice $\widetilde{L}(U, A, \widetilde{I})$ is complete lattice.

B. Two-Way Learning in Fuzzy Formal Context

For a fuzzy formal context $(U, A, \widetilde{I})$, 2^U and 2^A be the power set of U and A, respectively. The $\widetilde{\mathcal{F}} : 2^U \to 2^A$ and $\mathcal{H} : 2^A \to 2^U$ are considered as two set valued mappings, and they are abbreviated as $\widetilde{\mathcal{F}}$ and $\mathcal{H}$, respectively.

Let $(U, A, \widetilde{I})$ be a fuzzy formal context, L_1 and $\widetilde{L}_2$ be complete lattices and fuzzy complete lattices, respectively. The operators $(*, *)$ defined by the basic notions in fuzzy context are the cognitive operator $\widetilde{\mathcal{F}}$ and $\mathcal{H}$ of $(U, A, \widetilde{I})$, respectively, where complete lattice $L_1 = (\cap, \cup, \sim, U)$ and fuzzy complete lattice $\widetilde{L}_2 = (\wedge, \vee, \sim, \Gamma^A)$.

In what follows, we do not distinguish operator symbol representation from $(*, *)$ and $(\widetilde{\mathcal{F}}, \mathcal{H})$. Let $\widetilde{L}$ be a lattice, where $O_{\widetilde{L}}$ and $1_{\widetilde{L}}$ are zero and the unit element, respectively.

Let L_1 and $\widetilde{L}_2$ be a pair of complete lattices, for any $X_1, X_2 \in L_1, \widetilde{\mathcal{F}} : L_1 \to \widetilde{L}_2$ is an extent-intent cognitive operator if $\widetilde{\mathcal{F}}$ satisfies the following:

1) $\widetilde{\mathcal{F}}(0_{L_1}) = 1_{\widetilde{L}_2}, \widetilde{\mathcal{F}}(1_{L_1}) = 0_{\widetilde{L}_2}$;
2) $\widetilde{\mathcal{F}}(X_1 \vee X_2) = \widetilde{\mathcal{F}}(X_1) \wedge \widetilde{\mathcal{F}}(X_2)$.

Similarly, for any $\widetilde{B}_1, \widetilde{B}_2 \in \widetilde{L}_2, \mathcal{H} : \widetilde{L}_2 \to L_1$ is an extent-intent cognitive operator if

1) $\mathcal{H}(0_{\widetilde{L}_2}) = 1_{L_1}, \mathcal{H}(1_{\widetilde{L}_2}) = 0_{L_1}$;
2) $\mathcal{H}(\widetilde{B}_1 \vee \widetilde{B}_2) = \mathcal{H}(\widetilde{B}_1) \wedge \mathcal{H}(\widetilde{B}_2)$.

Definition 1 (See [25]): Let $(U, A, \widetilde{I})$ be a fuzzy formal context, L_1 and $\widetilde{L}_2$ be two complete lattices, $\widetilde{\mathcal{F}}$ and H be two cognitive operators (i.e.,$(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$ is a cognitive system). For any $X \in L_1$, $\widetilde{B} \in \widetilde{L}_2$, denote

$$\widetilde{\mathcal{G}}_1 = \{(X, \widetilde{B}) | \widetilde{B} \leqslant \widetilde{\mathcal{F}}(X), X \leqslant \mathcal{H}(\widetilde{B})\}$$

$$\widetilde{\mathcal{G}}_2 = \{(X, \widetilde{B}) | \widetilde{\mathcal{F}}(X) \leqslant \widetilde{B}, \mathcal{H}(\widetilde{B}) \leqslant X\}.$$

1) If $(X, \widetilde{B}) \in \widetilde{\mathcal{G}}_1$, then $(X, \widetilde{B})$ is a necessary fuzzy granule concept of $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$. Meanwhile, $\widetilde{\mathcal{G}}_1$ is a necessary fuzzy granule concept space of $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$.
2) If $(X, \widetilde{B}) \in \widetilde{\mathcal{G}}_2$, then $(X, \widetilde{B})$ is a sufficient fuzzy granule concept of $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$. Meanwhile, $\widetilde{\mathcal{G}}_2$ is a sufficient fuzzy granule concept space of $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$.
3) If $(X, \widetilde{B}) \in \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$, that is, $(X, \widetilde{B})$ satisfies $\widetilde{B} = \widetilde{\mathcal{F}}(X)$ and $X = \mathcal{H}(\widetilde{B})$, then $(X, \widetilde{B})$ is a sufficient and necessary fuzzy granule concept of $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$. Meanwhile, $\widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$ is a sufficient and necessary fuzzy granule concept space of $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$.

From Definition 1, we only consider the situation that there exist three fuzzy granule concept spaces in $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$. Note that $\widetilde{\mathcal{G}}_1 \cup \widetilde{\mathcal{G}}_2$ be a fuzzy granule concept space. Therefore, $\widetilde{\mathcal{G}}_1 \cup \widetilde{\mathcal{G}}_2$ is the concept space of $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$. However, $(X, \widetilde{B}) \notin \widetilde{\mathcal{G}}_1 \cup \widetilde{\mathcal{G}}_2$ is not a fuzzy granule concept of $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$. Moreover, if necessary, sufficient, sufficient and necessary fuzzy granule concepts do not exist at the beginning of $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$. The approaches to the evolution of these fuzzy granule concepts are as follows.

Property 1 (See [19]): Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$ be a cognitive system, $\widetilde{\mathcal{G}}_1$ be a necessary fuzzy granule concept and $\widetilde{\mathcal{G}}_2$ be a sufficient fuzzy granule concept. If $X \in L_1, \widetilde{B} \in \widetilde{L}_2$, then

1) $(X \wedge \mathcal{H}(\widetilde{B}), \widetilde{B} \vee \widetilde{\mathcal{F}}(X)) \in \widetilde{\mathcal{G}}_1$;
2) $(X \vee \mathcal{H}(\widetilde{B}), \widetilde{B} \wedge \widetilde{\mathcal{F}}(X)) \in \widetilde{\mathcal{G}}_1$;
3) $(\mathcal{H}(\widetilde{B}), \widetilde{B} \wedge \widetilde{\mathcal{F}}(X)) \in \widetilde{\mathcal{G}}_1$;
4) $(X \wedge \mathcal{H}(\widetilde{B}), \widetilde{\mathcal{F}}(X)) \in \widetilde{\mathcal{G}}_1$;
5) $(\mathcal{H}\widetilde{\mathcal{F}}(X), \widetilde{B} \wedge \widetilde{\mathcal{F}}(X)) \in \widetilde{\mathcal{G}}_1$;
6) $(X \wedge \mathcal{H}(\widetilde{B}), \widetilde{\mathcal{F}}\mathcal{H}(\widetilde{B})) \in \widetilde{\mathcal{G}}_1$;
7) $(X \vee \mathcal{H}(\widetilde{B}), \widetilde{\mathcal{F}}\mathcal{H}(\widetilde{B})) \in \widetilde{\mathcal{G}}_2$;
8) $(\mathcal{H}\widetilde{\mathcal{F}}(X), \widetilde{B} \vee \widetilde{\mathcal{F}}(X)) \in \widetilde{\mathcal{G}}_2$.

Property 2 (See [25]): Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$ be a cognitive system, $\widetilde{\mathcal{G}}_1$ be a necessary fuzzy granule concept space and $\widetilde{\mathcal{G}}_2$ be a sufficient fuzzy granule concept space. If $(X_1, \widetilde{B}_1) \in \widetilde{\mathcal{G}}_1$ and $(X_2, \widetilde{B}_2) \in \widetilde{\mathcal{G}}_2$, then

1) $(X_1 \vee \mathcal{H}(\widetilde{B}_1), \widetilde{\mathcal{F}}(X_1 \vee \mathcal{H}(\widetilde{B}_1))) \in \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$
2) $(\mathcal{H}(\widetilde{B}_1 \vee \widetilde{\mathcal{F}}(X_1)), \widetilde{B}_1 \vee \widetilde{\mathcal{F}}(X_1)) \in \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$
3) $(X_2 \wedge \mathcal{H}(\widetilde{B}_2), \widetilde{\mathcal{F}}(X_2 \wedge \mathcal{H}(\widetilde{B}_2))) \in \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$
4) $(\mathcal{H}(\widetilde{B}_2 \wedge \widetilde{\mathcal{F}}(X_2)), \widetilde{B}_2 \wedge \widetilde{\mathcal{F}}(X_2)) \in \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$.

Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$ be a cognitive system, $\widetilde{\mathcal{G}}_1$ be a necessary fuzzy granule concept space, and $\widetilde{\mathcal{G}}_2$ be a sufficient fuzzy granule concept space. If $''\vee''$ and $''\wedge''$ are defined operators of cognitive system, and

1) $(X_1, \widetilde{B}_1) \wedge (X_2, \widetilde{B}_2) = (X_1 \wedge X_2, \widetilde{\mathcal{F}}\mathcal{H}(\widetilde{B}_1 \vee \widetilde{B}_2))$;
2) $(X_1, \widetilde{B}_1) \vee (X_2, \widetilde{B}_2) = (\mathcal{H}\widetilde{\mathcal{F}}(X_1 \vee X_2), \widetilde{B}_1 \wedge \widetilde{B}_2)$.

Therefore, we can easily find some problems in two-way learning as follows:

1) the sufficient and necessary fuzzy granule concept have to obtain from the necessary or sufficient fuzzy granule concept;
2) two-way learning cannot integrate past experiences into itself to deal with dynamic data.

A novel cognitive mechanism based on two-way learning in a fuzzy formal context is introduced as follows based on the problem mentioned previously in this article.

C. Granule Concept Learning

Let $\widetilde{\mathcal{F}}$ and $\mathcal{H}$ be two cognitive operators. For any $x \in U$ and $a \in A$, we say that $(\mathcal{H}\widetilde{\mathcal{F}}(x), \widetilde{\mathcal{F}}(x))$ and $(\mathcal{H}(a), \widetilde{\mathcal{F}}\mathcal{H}(a))$ are fuzzy granule concepts.

Definition 2 (see [20]): Let U_{i-1} and U_i be object sets of $\{U_t\}\uparrow$ and A_{i-1} and A_i be attribute sets of $\{A_t\}\uparrow$, where $\{U_t\}\uparrow$ is a nondecreasing sequence subset of U, that is, $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n$; $\{A_t\}\uparrow$ is a nondecreasing sequence subset of A, that is, $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m$; Denote $\Delta U_{i-1} = U_i - U_{i-1}$ and $\Delta A_{i-1} = A_i - A_{i-1}$. Suppose

1) $\widetilde{\mathcal{F}}_{i-1} : 2^{U_{i-1}} \to 2^{A_{i-1}}$, $\quad \mathcal{H}_{i-1} : 2^{A_{i-1}} \to 2^{U_{i-1}}$;
2) $\widetilde{\mathcal{F}}_{\Delta U_{i-1}} : 2^{\Delta U_{i-1}} \to 2^{A_{i-1}}$, $\mathcal{H}_{\Delta U_{i-1}} : 2^{A_{i-1}} \to 2^{\Delta U_{i-1}}$;
3) $\widetilde{\mathcal{F}}_{\Delta A_{i-1}} : 2^{U_i} \to 2^{A_{i-1}}$, $\quad \mathcal{H}_{\Delta A_{i-1}} : 2^{A_{i-1}} \to 2^{U_i}$;
4) $\widetilde{\mathcal{F}}_i : 2^{U_i} \to 2^{A_i}$, $\quad \mathcal{H}_i : 2^{A_i} \to 2^{U_i}$

are four pairs of cognitive operators satisfying the following properties:

$$\widetilde{\mathcal{F}}(x) = \begin{cases} \widetilde{\mathcal{F}}_{i-1}(x) \cup \widetilde{\mathcal{F}}_{\Delta A_{i-1}}(x), & \text{if } x \in U_{i-1} \\ \widetilde{\mathcal{F}}_{\Delta U_{i-1}}(x) \cup \widetilde{\mathcal{F}}_{\Delta A_{i-1}}(x), & \text{otherwise} \end{cases}$$

$$\mathcal{H}(a) = \begin{cases} \mathcal{H}_{i-1}(a) \cup \mathcal{H}_{\Delta U_{i-1}}(a), & \text{if } x \in U_{i-1} \\ \mathcal{H}_{\Delta A_{i-1}}(a), & \text{otherwise} \end{cases}$$

then we say that $\widetilde{\mathcal{F}}_i$ and $\mathcal{H}_i$ are extended cognitive operators of $\widetilde{\mathcal{F}}_{i-1}$ and $\mathcal{H}_{i-1}$ with the update information $\widetilde{\mathcal{F}}_{\Delta U_{i-1}}$, $\mathcal{H}_{\Delta U_{i-1}}$ and $\widetilde{\mathcal{F}}_{\Delta A_{i-1}}$, $\mathcal{H}_{\Delta A_{i-1}}$.

III. Fuzzy-Based Two-Way Concept-Cognitive Learning (F-TCCL)

In this section, the following two tasks need to be done to verify our proposed method:

1) analyze the relationship of fuzzy granule concepts in a two-way learning system;
2) explore a novel concept learning mechanism for the proposed TCCL.

A. Fuzzy-Based Two-Way Granule Concept

Unlike other two-way learning systems [19], [25], [26], F-TCCL is more flexible and less time-consuming to learn concepts from the given clue. Moreover, we verify that the number of sufficient and necessary fuzzy granule concepts is no more than 6, no more than 2 in special cases. Furthermore, we investigate a mechanism for directly learning sufficient and necessary fuzzy granule concepts (i.e., fuzzy concepts).

Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$ be a cognitive system, $\widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$ be a sufficient and necessary fuzzy granule concept space. The fuzzy granule concept $(X, \widetilde{B})$ is a sufficient and necessary fuzzy granule concept, if any $X \in L_1$ and $\widetilde{B} \in \widetilde{L}_2$, $(X, \widetilde{B}) \in \widetilde{\mathcal{G}}_1 \cup \widetilde{\mathcal{G}}_2$, then

1) $(\mathcal{H}\widetilde{\mathcal{F}}(X), \widetilde{\mathcal{F}}(X)) \in \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$;
2) $(\mathcal{H}(\widetilde{B}), \widetilde{\mathcal{F}}\mathcal{H}(\widetilde{B})) \in \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$.

Note that $(\mathcal{H}\widetilde{\mathcal{F}}(X), \widetilde{\mathcal{F}}(X))$ and $(\mathcal{H}(\widetilde{B}), \widetilde{\mathcal{F}}\mathcal{H}(\widetilde{B}))$ are two fuzzy granule concepts of $\widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$. Next, we will examine the specific relationship between them.

Proposition 1: Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$ be a cognitive system, $\widetilde{\mathcal{G}}_1$ be a necessary fuzzy granule concept space, $\widetilde{\mathcal{G}}_2$ be a sufficient fuzzy granule concept space. For arbitrary information granule $(X, \widetilde{B}) \in \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$, there is only one sufficient and necessary fuzzy granule concept is itself (i.e., $(X, \widetilde{B})$); Otherwise, we have two sufficient and necessary fuzzy granule concepts, that is, $(\mathcal{H}\widetilde{\mathcal{F}}(X), \widetilde{\mathcal{F}}(X))$ and $(\mathcal{H}(\widetilde{B}), \widetilde{\mathcal{F}}\mathcal{H}(\widetilde{B}))$.

Proof: To prove this proposition, we divide it into the following two steps.

1) For one sufficient and necessary fuzzy granule concept. Because $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$ be a cognitive system, it is immediate from Properties 1 and 2.
2) For two sufficient and necessary fuzzy granule concepts. Because $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$ be a cognitive system, from Definition 1 and Property 1, we have three type spaces: $\widetilde{\mathcal{G}}_1$, $\widetilde{\mathcal{G}}_2$ and $(\widetilde{\mathcal{G}}_1 \cup \widetilde{\mathcal{G}}_2)^c$, where $(\cdot)^c$ is the complement. Then, we divide it into three cases to prove it.
   a) If $(X, \widetilde{B}) \in \widetilde{\mathcal{G}}_1$, from Definition 1, we have $X \leqslant \mathcal{H}(\widetilde{B})$ and $\widetilde{B} \leqslant \widetilde{\mathcal{F}}(X)$. Thus, $X \vee \mathcal{H}(\widetilde{B}) = \mathcal{H}(\widetilde{B})$, $\widetilde{\mathcal{F}}(X \vee \mathcal{H}(\widetilde{B})) = \widetilde{\mathcal{F}}\mathcal{H}(\widetilde{B})$; $\widetilde{B} \vee \widetilde{\mathcal{F}}(X) = \widetilde{\mathcal{F}}(X)$, $\mathcal{H}(\widetilde{B} \vee \widetilde{\mathcal{F}}(X)) = \mathcal{H}\widetilde{\mathcal{F}}(X)$; Hence, two sufficient and necessary fuzzy granule concepts are $(\mathcal{H}\widetilde{\mathcal{F}}(X), \widetilde{\mathcal{F}}(X))$ and $(\mathcal{H}(\widetilde{B}), \widetilde{\mathcal{F}}\mathcal{H}(\widetilde{B}))$.
   b) If $(X, \widetilde{B}) \in \widetilde{\mathcal{G}}_2$, from Definition 1, we have $\widetilde{\mathcal{F}}(X) \leqslant \widetilde{B}$ and $\mathcal{H}(\widetilde{B}) \leqslant X$. Thus, $X \wedge H(\widetilde{B}) = H(\widetilde{B})$, $\widetilde{\mathcal{F}}(X \wedge \mathcal{H}(\widetilde{B})) = \widetilde{\mathcal{F}}H(\widetilde{B})$; $\widetilde{B} \wedge \widetilde{\mathcal{F}}(X) = \mathcal{H}(\widetilde{B})$, $\mathcal{H}(\widetilde{B} \wedge \widetilde{\mathcal{F}}(X)) = \mathcal{H}\widetilde{\mathcal{F}}(X)$; hence, two sufficient and necessary fuzzy granule concepts are $(\mathcal{H}\widetilde{\mathcal{F}}(X), \widetilde{\mathcal{F}}(X))$ and $(\mathcal{H}(\widetilde{B}), \widetilde{\mathcal{F}}\mathcal{H}(\widetilde{B}))$.
   c) If $(X, \widetilde{B}) \notin \widetilde{\mathcal{G}}_1 \cup \widetilde{\mathcal{G}}_2$, it is immediate from Definition 1.

By combining 1) and 2), this proposition is proven.

Corollary 1: Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$ be a cognitive system, $\widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$ be a sufficient and necessary fuzzy granule concept space. $(X, \widetilde{B}) = (\mathcal{H}\widetilde{\mathcal{F}}(X), \widetilde{\mathcal{F}}(X)) = (\mathcal{H}(\widetilde{B}), \widetilde{\mathcal{F}}\mathcal{H}(\widetilde{B}))$, if any $X \in L_1$ and $\widetilde{B} \in \widetilde{L}_2$, $(X, \widetilde{B})$ is sufficient and necessary fuzzy granule concept.

From the aforementioned analysis, we can notice the following two conclusions:

1) sufficient and necessary fuzzy granule concept can be obtained directly from arbitrary information granules;
2) for arbitrary information granule $(X, \widetilde{B})$, the number of sufficient and necessary fuzzy granule concepts is no more than six through a two-way learning system.

Based on them, we show some new notions and propositions in the following subsection.

B. Learning Mechanism

Similar to the initial information X and $\widetilde{B}$, $X^{\widetilde{\mathcal{F}}}$ and $\widetilde{B}^{\mathcal{H}}$ also represent extent and intent. We need to point out that $X^{\widetilde{\mathcal{F}}}$ and $\widetilde{B}^{\mathcal{H}}$ represent no more than one cognitive operation $\widetilde{\mathcal{F}}$ or $\mathcal{H}$, respectively. It is easy to prove that $X^{\widetilde{\mathcal{F}}} \in \{X, X \vee \mathcal{H}(\widetilde{B}), X \wedge \mathcal{H}(\widetilde{B})\}$ and $\widetilde{B}^{\mathcal{H}} \in \{\widetilde{B}, \widetilde{B} \vee \widetilde{\mathcal{F}}(X), \widetilde{B} \wedge \widetilde{\mathcal{F}}(X)\}$ according to $\widetilde{\mathcal{G}}_1$ and $\widetilde{\mathcal{G}}_2$ in Property 1, where $X \in L_1, \widetilde{B} \in \widetilde{L}_2$.

Note that $(X^{\widetilde{\mathcal{F}}}, \widetilde{B}^{\mathcal{H}})$ is the fuzzy granule concept obtain through cognitive learning in the cognitive system $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$, where $X \in L_1$ and $\widetilde{B} \in \widetilde{L}_2$. Thus, we have a novel cognitive mechanism of sufficient and necessary fuzzy granule concept in the following statements hold.

Proposition 2: Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$ be a cognitive system, $\widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2 = \{(\mathcal{H}\widetilde{\mathcal{F}}(X^{\widetilde{\mathcal{F}}}), \widetilde{\mathcal{F}}(X^{\widetilde{\mathcal{F}}})) | X \in L_1\} \cup \{(\mathcal{H}(\widetilde{B}^{\mathcal{H}}), \widetilde{\mathcal{F}}\mathcal{H}(\widetilde{B}^{\mathcal{H}})) | \widetilde{B} \in \widetilde{L}_2\}$ be a sufficient and necessary fuzzy granule concept space. Then, the following statements hold:

1) $(\mathcal{H}\widetilde{\mathcal{F}}(X), \widetilde{\mathcal{F}}(X)) \in \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$;
2) $(\mathcal{H}\widetilde{\mathcal{F}}(X \wedge \mathcal{H}(\widetilde{B})), \widetilde{\mathcal{F}}(X \wedge \mathcal{H}(\widetilde{B}))) \in \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$;
3) $(\mathcal{H}\widetilde{\mathcal{F}}(X \vee \mathcal{H}(\widetilde{B})), \widetilde{\mathcal{F}}(X \vee \mathcal{H}(\widetilde{B}))) \in \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$;
4) $(\mathcal{H}(\widetilde{B}), \widetilde{\mathcal{F}}\mathcal{H}(\widetilde{B})) \in \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$;
5) $(\mathcal{H}(\widetilde{B} \vee \widetilde{\mathcal{F}}(X)), \widetilde{\mathcal{F}}\mathcal{H}(\widetilde{B} \vee \widetilde{\mathcal{F}}(X))) \in \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$;
6) $(\mathcal{H}(\widetilde{B} \wedge \widetilde{\mathcal{F}}(X)), \widetilde{\mathcal{F}}\mathcal{H}(\widetilde{B} \wedge \widetilde{\mathcal{F}}(X))) \in \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$.

Proof: The proof can be derived by Definition 1 and Proposition 1. ■

From the aforementioned discussion, we know that the cognitive process of sufficient and necessary fuzzy granule concept is learning more knowledge from the unknown via a pair of cognitive operators (i.e., extent-intent and intent-extent). Hence, a pair of cognitive operations $\mathcal{H}$ and $\widetilde{\mathcal{F}}$ are used to learn the sufficient and necessary fuzzy granule concept in our method. Based on the cognitive operators $\mathcal{H}$ and $\widetilde{\mathcal{F}}$ in Definition 1, we give six learning methods of sufficient and necessary fuzzy granule concepts shown in Proposition 2.

Corollary 2: Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$ be a cognitive system, $\widetilde{\mathcal{F}}$ and $\mathcal{H}$ be a pair of cognitive operators. For $\widetilde{\mathcal{G}}_1 \cup \widetilde{\mathcal{G}}_2$ is a finite nonempty fuzzy granule concept space. A weak tripartition $\widetilde{\mathcal{G}} = \{\widetilde{\mathcal{G}}_1, \widetilde{\mathcal{G}}_2, \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2\}$. For $(X^{\widetilde{\mathcal{F}}}, \widetilde{B}^{\mathcal{H}}) \in \widetilde{\mathcal{G}}_1 \cup \widetilde{\mathcal{G}}_2$, we have

1) $\widetilde{\mathcal{G}}_1 = \{(X^{\widetilde{\mathcal{F}}}, \widetilde{B}^{\mathcal{H}}) | \widetilde{B}^{\mathcal{H}} \leqslant \widetilde{\mathcal{F}}(X^{\widetilde{\mathcal{F}}}), X^{\widetilde{\mathcal{F}}} \leqslant \mathcal{H}(\widetilde{B}^{\mathcal{H}})\}$;
2) $\widetilde{\mathcal{G}}_2 = \{(X^{\widetilde{\mathcal{F}}}, \widetilde{B}^{\mathcal{H}}) | \widetilde{\mathcal{F}}(X^{\widetilde{\mathcal{F}}}) \leqslant \widetilde{B}^{\mathcal{H}}, H(\widetilde{B}^{\mathcal{H}}) \leqslant X^{\widetilde{\mathcal{F}}}\}$;
3) $\widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2 = \{(X^{\widetilde{\mathcal{F}}}, \widetilde{B}^{\mathcal{H}}) | \widetilde{B}^{\mathcal{H}} = \widetilde{\mathcal{F}}(X^{\widetilde{\mathcal{F}}}), X^{\widetilde{\mathcal{F}}} = \mathcal{H}(\widetilde{B}^{\mathcal{H}})\}$.

According to the aforementioned discussion, one can directly learn sufficient and necessary fuzzy granule concepts from arbitrary information granules through the learning mechanism based on F-TCCL. The details as shown in Algorithm 1.

Algorithm 1: Learning Fuzzy Granule Concept.

Input: Arbitrary information granule $(X, \widetilde{B})$, a fuzzy formal context $(U, A, \widetilde{I})$, fuzzy granule concept space $\widetilde{\mathcal{G}}_3$.
Output: Fuzzy granule concept: $(X^3, \widetilde{B}^3)$.

```
Let G̃3 = ∅;
while (X, B̃) is true do
    Learn fuzzy granule concept (X^3, B̃^3) from (X, B̃),
    (X^3_1, B̃^3_1), ..., (X^3_m, B̃^3_m), m ⩽ 6, by six methods
    according to Proposition 2;
    G̃3 ← (X^3, B̃^3).
end while
```

Given a fuzzy formal context $F=(U,A,\widetilde{I})$. The cardinality of objects and attributes are denoted by $|U|$ and $|A|$, respectively. The cardinality of objects and attributes of arbitrary information granule $(X,\widetilde{B})$ is denoted by $|X|$ and $|\widetilde{B}|$. Next, we can analyze the time complexity of Algorithm 1. Running step 1 takes $O(1)$ due to initialized setting. The while-statement decides its running time in steps 2–5. Thus, the running time complexity of Algorithm 1 takes $O(|U||A|(|X|+|\widetilde{B}|))$.

The primary motivation of F-TCCL is to provide an idea for cross-level learning for sufficient and necessary fuzzy granule concept space, as shown in Fig. 2. Unlike the two-way learning system, it does not need to thoroughly learn the necessary, sufficient fuzzy granule concept, although it emphasizes that the learning mechanism should be two-way.

IV. Progressive Cognitive Learning of F-TCCL

Based on the aforementioned discussion, in this section, we put forward the F-TCCL with progressive learning (P-FTCCL), which can perform well compared with some two-way learning systems [19], [25], [26] in terms of learning concepts and saving time. Furthermore, we explore two update mechanisms for the proposed P-FTCCL.

A. Granulation Relation

Note that Zhang and Xu [18] first investigated learning concepts based on granular computing, and Xu et al. [19] proposed a novel cognitive system for learning concepts via the transformation of information granules. Particularly worth mentioning is that granular computing provides an effective method for CCL and many studies [20], [25], [28] show that the framework is worthwhile. For convenience, we use $a\in\widetilde{B}$ to represent the attribute a in the attribute set defined by fuzzy set $\widetilde{B}$ when no confusion exists.

Given $(L_1,\widetilde{L}_2,\widetilde{\mathcal{F}},\mathcal{H})$ be a cognitive system. For any $X\in L_1$, $\widetilde{B}\in\widetilde{L}_2$, we have $\widetilde{\mathcal{F}}(X)=\bigcap_{x\in X}\widetilde{\mathcal{F}}(x)$, $\mathcal{H}(\widetilde{B})=\bigcap_{a\in\widetilde{B}}\mathcal{H}(a)$. Then, the following statements hold:

$$(X,\widetilde{B})=\bigvee_{x\in X}(\mathcal{H}\widetilde{\mathcal{F}}(x),\widetilde{\mathcal{F}}(x))=\bigwedge_{a\in\widetilde{B}}(\mathcal{H}(a),\widetilde{\mathcal{F}}\mathcal{H}(a)).$$

Definition 3: Let $(L_1,\widetilde{L}_2,\widetilde{\mathcal{F}},\mathcal{H})$ be a cognitive system, $\widetilde{\mathcal{F}}$ and $\mathcal{H}$ be a pair of cognitive operators, $\widetilde{\mathcal{G}}_1\cup\widetilde{\mathcal{G}}_2$ be a finite nonempty fuzzy granule concept space. For any $(X^{\widetilde{\mathcal{F}}},\widetilde{B}^{\mathcal{H}})\in\widetilde{\mathcal{G}}_1\cup\widetilde{\mathcal{G}}_2$, we have

$$\widetilde{\mathcal{G}}_1\cap\widetilde{\mathcal{G}}_2=\left\{\bigvee_{x\in X^{\widetilde{\mathcal{F}}}}(\mathcal{H}\widetilde{\mathcal{F}}(x),\widetilde{\mathcal{F}}(x))|X\in L_1\right\}$$
$$\cup\left\{\bigwedge_{a\in\widetilde{B}^{\mathcal{H}}}(\mathcal{H}(a),\widetilde{\mathcal{F}}\mathcal{H}(a))|\widetilde{B}\in\widetilde{L}_2\right\}$$

where $X^{\widetilde{\mathcal{F}}}$ and $\widetilde{B}^{\mathcal{H}}$ represent no more than one cognitive operation (i.e., $\widetilde{\mathcal{F}}$ and $\mathcal{H}$).

Considering that fuzzy granule concept in CCL is an effective method to build a concept space [20], [22], [23], [28], we can obtain a new formalized method of F-TCCL based on the granulation mechanism as follows.

Proposition 3: Let $(L_1,\widetilde{L}_2,\widetilde{\mathcal{F}},\mathcal{H})$ be a cognitive system, and $\widetilde{\mathcal{G}}_1\cap\widetilde{\mathcal{G}}_2$ be a sufficient and necessary fuzzy granule concept space. For any $\left(X^{\widetilde{\mathcal{F}}},\widetilde{B}^{\mathcal{H}}\right)\in\widetilde{\mathcal{G}}_1\cup\widetilde{\mathcal{G}}_2$, the following statements hold:

1) $\left(\bigcap_{a\in\widetilde{\mathcal{F}}(X^{\widetilde{\mathcal{F}}})}\mathcal{H}(a),\ \bigcap_{x\in X^{\widetilde{\mathcal{F}}}}\widetilde{\mathcal{F}}(x)\right)\in\widetilde{\mathcal{G}}_1\cap\widetilde{\mathcal{G}}_2$;

2) $\left(\bigcap_{a\in\widetilde{B}^{\mathcal{H}}}\mathcal{H}(a),\ \bigcap_{x\in\mathcal{H}}(\widetilde{B}^{\mathcal{H}})\widetilde{\mathcal{F}}(x)\right)\in\widetilde{\mathcal{G}}_1\cap\widetilde{\mathcal{G}}_2$.

Proof: It is immediate from Proposition 2 and Definition 3. ■

Intrinsically, a two-way learning mechanism is the cognitive systems begin to acquire concepts from the clues. It is not difficult to find that two-way learning is an influential way to learn the concept from arbitrary clues but is unsuitable for dynamic data. Thus, this subsection explores the granulation relation of F-TCCL. Next, we consider how the sufficient and necessary fuzzy granule concept space is updated with the newly input information in the fuzzy formal context.

B. Progressive Learning by Granulation Mechanism

As described in Proposition 2 of Section III-B, we began to learn fuzzy granule concepts from a given clue via a TCCL system. In the learning process, we can learn the concept from the arbitrary object set X or fuzzy set $\widetilde{B}$. Of course, we can also learn the concept from an information granule $(X,\widetilde{B})$. That is to say that we have the update mechanisms for F-TCCL (i.e., object-oriented concept learning and attribute-oriented concept learning).

According to the analysis in Section IV-A, one can learn a fuzzy granule concept space from arbitrary information granule $(X,\widetilde{B})$ and cognitive operators (i.e., $\widetilde{\mathcal{F}}$, $\mathcal{H}$) through granulation mechanism of concept (i.e., $\widetilde{\mathcal{F}}(x)$, $\mathcal{H}(a)$). Then, we propose a progressive learning process by granulation relation of F-TCCL. Considering the information on the object set and attribute set will be updated as time goes by in the real world, especially the information in the given clues. Therefore, hereinafter, we only discuss the updated object set while the attributes in the clue are not updated, and the attribute set is updated while the samples in the clue are not updated.

Definition 4: Let U_{i-1} and U_i be object sets of $\{U_t\}\uparrow$, A_{i-1} and A_i be attribute sets of $\{A_t\}\uparrow$, where $\{U_t\}\uparrow$ is a nondecreasing sequence subset of U, that is, $U_1\subseteq U_2\subseteq\cdots\subseteq U_n$; $\{A_t\}\uparrow$ is a nondecreasing sequence subset of A, that is, $A_1\subseteq A_2\subseteq\cdots\subseteq A_m$; let X_{i-1} and X_i be object sets of $\{X_t\}\uparrow$, $\widetilde{B}_{i-1}$ and $\widetilde{B}_i$ be fuzzy set of $\{\widetilde{B}_t\}\uparrow$, where $\{X_t\}\uparrow$ is a nondecreasing sequence subset of X, that is, $X_1\subseteq X_2\subseteq\cdots\subseteq X_n$; $\{\widetilde{B}_t\}\uparrow$ is a nondecreasing sequence subset of $\widetilde{B}$, that is, $\widetilde{B}_1\subseteq\widetilde{B}_2\subseteq\cdots\subseteq\widetilde{B}_m$; Denote $\Delta U_{i-1}=U_i-U_{i-1}$, $\Delta A_{i-1}=A_i-A_{i-1}$, $\Delta X_{i-1}=X_i-X_{i-1}$, and $\Delta\widetilde{B}_{i-1}=\widetilde{B}_i-\widetilde{B}_{i-1}$. Suppose

1) $\widetilde{\mathcal{F}}_{i-1}: 2^{X_{i-1}} \to 2^{A_{i-1}}, \quad \mathcal{H}_{i-1}: 2^{\widetilde{B}_{i-1}} \to 2^{U_{i-1}}$;
2) $\widetilde{\mathcal{F}}_{\Delta X_{i-1}}: 2^{\Delta X_{i-1}} \to 2^{A_{i-1}}, \mathcal{H}_{\Delta X_{i-1}}: 2^{\widetilde{B}_{i-1}} \to 2^{\Delta U_{i-1}}$;
3) $\widetilde{\mathcal{F}}_{\Delta \widetilde{B}_{i-1}}: 2^{X_i} \to 2^{\Delta A_{i-1}}, \quad \mathcal{H}_{\Delta \widetilde{B}_{i-1}}: 2^{\Delta \widetilde{B}_{i-1}} \to 2^{U_i}$;
4) $\widetilde{\mathcal{F}}_i: 2^{X_i} \to 2^{A_i}, \quad \mathcal{H}_i: 2^{\widetilde{B}_i} \to 2^{U_i}$

are four pairs of cognitive operators satisfying the following properties:

$$\widetilde{\mathcal{F}}_i(x) = \begin{cases} \widetilde{\mathcal{F}}_{i-1}(x) \cup \widetilde{\mathcal{F}}_{\Delta \widetilde{B}_{i-1}}(x), & \text{if } (1) \\ \widetilde{\mathcal{F}}_{i-1}(x) \cap \widetilde{\mathcal{F}}_{\Delta X_{i-1}}(x), & \text{if } (2) \\ \widetilde{\mathcal{F}}_{i-1}(x) \cap \widetilde{\mathcal{F}}_{\Delta X_{i-1}}(x) \cup \widetilde{\mathcal{F}}_{\Delta \widetilde{B}_{i-1}}(x), & \text{if } (3) \end{cases}$$

$$\mathcal{H}_i(a) = \begin{cases} \mathcal{H}_{i-1}(a) \cap \mathcal{H}_{\Delta \widetilde{B}_{i-1}}(a), & \text{if } (1) \\ \mathcal{H}_{i-1}(a) \cup \mathcal{H}_{\Delta X_{i-1}}(a), & \text{if } (2) \\ \mathcal{H}_{i-1}(a) \cap \mathcal{H}_{\Delta \widetilde{B}_{i-1}}(a) \cup \mathcal{H}_{\Delta X_{i-1}}(a), & \text{if } (3) \end{cases}$$

where (1) denote $\Delta X_{i-1} = \emptyset \& \Delta \widetilde{B}_{i-1} \neq \emptyset$, (2) denote $\Delta X_{i-1} \neq \emptyset \& \Delta \widetilde{B}_{i-1} = \emptyset$, (3) denote $\Delta X_{i-1} \neq \emptyset \& \Delta \widetilde{B}_{i-1} \neq \emptyset$; In addition, $\widetilde{\mathcal{F}}_{\Delta X_{i-1}}(x)$ and $\mathcal{H}_{\Delta X_{i-1}}(a)$ are empty set when $\Delta X_{i-1} = \emptyset$, and $\widetilde{\mathcal{F}}_{\Delta \widetilde{B}_{i-1}}(x)$ and $\mathcal{H}_{\Delta \widetilde{B}_{i-1}}(a)$ are empty set when $\Delta \widetilde{B}_{i-1} = \emptyset$.

Then, we say that $\widetilde{\mathcal{F}}_i$ and $\mathcal{H}_i$ are extent cognitive operators of $\widetilde{\mathcal{F}}_{i-1}$ and $\mathcal{H}_{i-1}$ with the update information $\widetilde{\mathcal{F}}_{\Delta X_{i-1}}$, $\widetilde{\mathcal{F}}_{\Delta \widetilde{B}_{i-1}}$, $\mathcal{H}_{\Delta X_{i-1}}$, and $\mathcal{H}_{\Delta \widetilde{B}_{i-1}}$. For convenience, the sufficient and necessary fuzzy granule concept space $\widetilde{\mathcal{G}}_3^i$ denotes the combination of $(\widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2)^1, (\mathcal{G}_1 \cap \widetilde{\mathcal{G}}_2)^2, \ldots, (\widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2)^n$. Moreover, it is pointed out that each update can be viewed as the outcome of updating the clues information under consideration once.

According to the discussion in Section III-B, given a cognitive system $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$, we obtain a new way to learn sufficient and necessary fuzzy granule concepts from given clues instead of necessary or sufficient fuzzy granule concepts. That is to say that we have six methods to learn fuzzy granule concepts [see items 1)–6) in Proposition 2]. Moreover, we obtain two update mechanisms in Section IV-C, which can also learn granule concept space in [20] and [28] by granulation mechanism.

However, in cognitive science, concept-cognitive was often considered incremental due to the whole being something else than the sum of its part [20], [31]. In what follows, by analyzing the update mechanism of TCCL, progressive learning by the granulation mechanism of F-TCCL is investigated in this subsection.

In this subsection, we first review the concept learning method proposed in Section III-B, and then, consider a novel concept learning method combined with the update mechanism in Section IV-C. The details are as follows.

Proposition 4: Given a fuzzy formal context $(U, A, \widetilde{I})$, Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}_i, \mathcal{H}_i)$ be a cognitive system with time series, ${\mathcal{G}_3}^i$ is the ith state sufficient and necessary fuzzy granule concept space. Then, the following statements hold:

1) $\left(\bigcap_{a \in \widetilde{\mathcal{F}}_i(X)} \mathcal{H}_i(a), \bigcap_{x \in X} \widetilde{\mathcal{F}}_i(x)\right) \in \widetilde{\mathcal{G}}_3^i$;

2) $\left(\bigcap_{a \in \widetilde{\mathcal{F}}_i(X \wedge \mathcal{H}_i(\widetilde{B}))} \mathcal{H}_i(a), \bigcap_{x \in X \wedge \mathcal{H}_i(\widetilde{B})} \widetilde{\mathcal{F}}_i(x)\right) \in \widetilde{\mathcal{G}}_3^i$;

3) $\left(\bigcap_{a \in \widetilde{\mathcal{F}}_i(X \vee \mathcal{H}_i(\widetilde{B}))} \mathcal{H}_i(a), \bigcap_{x \in X \vee \mathcal{H}_i(\widetilde{B})} \widetilde{\mathcal{F}}_i(x)\right) \in \widetilde{\mathcal{G}}_3^i$;

4) $\left(\bigcap_{a \in \widetilde{B}} \mathcal{H}_i(a), \bigcap_{x \in \mathcal{H}_i(\widetilde{B})} \widetilde{\mathcal{F}}_i(x)\right) \in \widetilde{\mathcal{G}}_3^i$;

5) $\left(\bigcap_{a \in \widetilde{B} \wedge \widetilde{\mathcal{F}}_i(X)} \mathcal{H}_i(a), \bigcap_{x \in \mathcal{H}_i(\widetilde{B} \wedge \widetilde{\mathcal{F}}_i(X))} \widetilde{\mathcal{F}}(x)\right) \in \widetilde{\mathcal{G}}_3^i$;

6) $\left(\bigcap_{a \in \widetilde{B} \vee \widetilde{\mathcal{F}}_i(X)} \mathcal{H}_i(a), \bigcap_{x \in \mathcal{H}_i(\widetilde{B} \vee \widetilde{\mathcal{F}}_i(X))} \widetilde{\mathcal{F}}_i(x)\right) \in \widetilde{\mathcal{G}}_3^i$.

where $\{X_t\} \uparrow$ is a nondecreasing sequence subset of X, and $\{\widetilde{B}_t\} \uparrow$ is a nondecreasing sequence subset of $\widetilde{B}$.

Proof: It is directly obtained from Proposition 3 and Definition 4. ■

C. Update Mechanism

According to the concept learning method in literature [22], we can learn a concept space from a fuzzy formal context. Essentially, the concept in the concept space is a sufficient and necessary fuzzy granule concept (i.e., $(\mathcal{H}\widetilde{\mathcal{F}}(x), \widetilde{\mathcal{F}}(x))$, $\mathcal{H}(a), (\widetilde{\mathcal{F}}\mathcal{H}(a))$. In this subsection, we study three update mechanisms of F-TCCL from a given clue.

Case 1: The update mechanism of object-oriented CCL.

Proposition 5: Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$ be a cognitive system, $\Delta X_{i-1} \neq \emptyset$ and $\Delta \widetilde{B}_{i-1} = \emptyset$, then the following statements hold:

1) $(\mathcal{H}_i \widetilde{\mathcal{F}}_i(X), \widetilde{\mathcal{F}}_i(X)) = \left(\bigcap_{a \in (\widetilde{\mathcal{F}}_{i-1}(X) \cap \widetilde{\mathcal{F}}_{\Delta X_{i-1}}(X))} (\mathcal{H}_{i-1}(a) \cup \mathcal{H}_{\Delta X_{i-1}}(a)), \widetilde{\mathcal{F}}_{i-1}(X) \cap \widetilde{\mathcal{F}}_{\Delta X_{i-1}}(X)\right)$;

2) $(\mathcal{H}_i(\widetilde{B}), \widetilde{\mathcal{F}}_i \mathcal{H}_i(\widetilde{B})) = \left(\mathcal{H}_{i-1}(\widetilde{B}) \cup \mathcal{H}_{\Delta X_{i-1}}(\widetilde{B}), \bigcap_{x \in \mathcal{H}_{i-1}(\widetilde{B}) \cup \mathcal{H}_{\Delta X_{i-1}}(\widetilde{B})} (\widetilde{\mathcal{F}}_{i-1}(x) \cap \widetilde{\mathcal{F}}_{\Delta X_{i-1}}(x))\right)$.

Proof: 1) According to Definition 4, we have $\widetilde{\mathcal{F}}(x)_{\Delta \widetilde{B}_{i-1}} = \emptyset$ and $\widetilde{\mathcal{F}}_i(x) = \widetilde{\mathcal{F}}_{i-1}(x) \cap \widetilde{\mathcal{F}}_{\Delta X_{i-1}}(x)$.

Thus, according to Proposition 3, we have $\widetilde{\mathcal{F}}_i(X) = \bigcap_{x \in X} (\widetilde{\mathcal{F}}_{i-1}(x) \cap \widetilde{\mathcal{F}}_{\Delta X_{i-1}}(x)) = (\bigcap_{x \in X_{i-1}} \widetilde{\mathcal{F}}_{i-1}(x)) \cap (\bigcap_{x \in \Delta X_{i-1}} \widetilde{\mathcal{F}}_{\Delta X_{i-1}}(x)) = \widetilde{\mathcal{F}}_{i-1}(X) \cap \widetilde{\mathcal{F}}_{\Delta X_{i-1}}(X)$.

Then, based on the discussion in Definition 4 and Proposition 4, we obtain $\mathcal{H}_i \widetilde{\mathcal{F}}_i(X) = \bigcap_{a \in \widetilde{\mathcal{F}}_{i-1}(X) \cap \widetilde{\mathcal{F}}_{\Delta X_{i-1}}(X)} \mathcal{H}_i(a) = \bigcap_{a \in \widetilde{\mathcal{F}}_{i-1}(X) \cap \widetilde{\mathcal{F}}_{\Delta X_{i-1}}(X)} (\mathcal{H}_{i-1}(a) \cup \mathcal{H}_{\Delta X_{i-1}}(a))$.

Thus, we have $(\mathcal{H}_i \widetilde{\mathcal{F}}_i(X), \widetilde{\mathcal{F}}_i(X)) = (\bigcap_{a \in (\widetilde{\mathcal{F}}_{i-1}(X) \cap \widetilde{\mathcal{F}}_{\Delta X_{i-1}}(X))} (\mathcal{H}_{i-1}(a) \cup \mathcal{H}_{\Delta X_{i-1}}(a)), \widetilde{\mathcal{F}}_{i-1}(X) \cap \widetilde{\mathcal{F}}_{\Delta X_{i-1}}(X))$.

2) Similarly, according to Definition 4, we have $\mathcal{H}_{\Delta \widetilde{B}_{i-1}}(\widetilde{B}) = \emptyset$ and $\mathcal{H}_i(\widetilde{B}) = \mathcal{H}_{i-1}(\widetilde{B}) \cup \mathcal{H}_{\Delta X_{i-1}}(\widetilde{B})$,

Algorithm 2: Update Mechanism of Object-Oriented CCL.

Input: Arbitrary information granule $(X, \widetilde{B})$, a fuzzy formal context $(U, A, \widetilde{I})$, fuzzy granule concept space $\widetilde{\mathcal{G}}_3$.
Output: Sufficient and necessary fuzzy granule concept space $\widetilde{\mathcal{G}}_3^n$.
1: Initialize $\widetilde{\mathcal{G}}_3^n = \{(\mathcal{H}_i\widetilde{\mathcal{F}}_i(X^{\widetilde{\mathcal{F}}}), \widetilde{\mathcal{F}}_i(X^{\widetilde{\mathcal{F}}}))|X \in L_1\} \cup \{(\mathcal{H}_i(\widetilde{B}^{\mathcal{H}}), \widetilde{\mathcal{F}}_i\mathcal{H}_i(\widetilde{B}^{\mathcal{H}}))|\widetilde{B} \in \widetilde{L}_2\}$
2: **while** $i \leqslant n$ **do**
3: **for all** $X \in L_1$ **do** learn sufficient and necessary fuzzy granule concept $(\mathcal{H}_i\widetilde{\mathcal{F}}_i(X), \widetilde{\mathcal{F}}_i(X))$ from $X^{\widetilde{\mathcal{F}}}$ according to the item 1) of Proposition 5
4: **end for**
5: **for all** $\widetilde{B} \in \widetilde{L}_2$ **do** learn sufficient and necessary fuzyy granule concept $(\mathcal{H}_i(\widetilde{B}), \widetilde{\mathcal{F}}_i\mathcal{H}_i(\widetilde{B}))$ from $\widetilde{B}^{\mathcal{H}}$ according to the item 2) of Proposition 5
6: **end for**
7: $i \leftarrow i + 1$
8: **end while**
9: Learn sufficient and necessary fuzzy granule concept space $\widetilde{\mathcal{G}}_3^i$ by six methods according to Proposition 4
10: **return** $\widetilde{\mathcal{G}}_3^i$

Algorithm 3: Update Mechanism of Attribute-Oriented CCL.

Input: Arbitrary information granule $(X, \widetilde{B})$, a formal context $(U, A, \widetilde{I})$, fuzzy granule concept space $\widetilde{\mathcal{G}}_3$.
Output: Sufficient and necessary fuzzy granule concept space $\widetilde{\mathcal{G}}_3^n$.
1: Initialize $\widetilde{\mathcal{G}}_3^n = \{(\mathcal{H}_i\widetilde{\mathcal{F}}_i(X^{\widetilde{\mathcal{F}}}), \widetilde{\mathcal{F}}_i(X^{\widetilde{\mathcal{F}}}))|X \in L_1\} \cup \{(\mathcal{H}_i(\widetilde{B}^{\mathcal{H}}), \widetilde{\mathcal{F}}_i\mathcal{H}_i(\widetilde{B}^{\mathcal{H}}))|\widetilde{B} \in \widetilde{L}_2\}$
2: **while** $i \leqslant n$ **do**
3: **for all** $X \in L_1$ **do** learn sufficient and necessary fuzzy granule concept $(\mathcal{H}_i\widetilde{\mathcal{F}}_i(X), \widetilde{\mathcal{F}}_i(X))$ from $X^{\widetilde{\mathcal{F}}}$ according to the item 1) of Proposition 6
4: **end for**
5: **for all** $\widetilde{B} \in \widetilde{L}_2$ **do** learn sufficient and necessary fuzzy granule concept $(\mathcal{H}_i(\widetilde{B}), \widetilde{\mathcal{F}}_i\mathcal{H}_i(\widetilde{B}))$ from $\widetilde{B}^{\mathcal{H}}$ according to the item 2) of Proposition 6
6: **end for**
7: $i \leftarrow i + 1$
8: **end while**
9: Learn sufficient and necessary fuzzy granule concept space $\widetilde{\mathcal{G}}_3^i$ by six methods according to Proposition 4
10: **return** $\widetilde{\mathcal{G}}_3^i$

then we have $\widetilde{\mathcal{F}}_i\mathcal{H}_i(\widetilde{B}) = \widetilde{\mathcal{F}}_i(\mathcal{H}_{i-1}(\widetilde{B}) \cup \mathcal{H}_{\Delta X_{i-1}}(\widetilde{B}) = \bigcap_{x\in\mathcal{H}_{i-1}(\widetilde{B})\cup\mathcal{H}_{\Delta X_{i-1}}(\widetilde{B})} \widetilde{\mathcal{F}}_i(x) = \bigcap_{x\in\mathcal{H}_{i-1}(\widetilde{B})\cup\mathcal{H}_{\Delta X_{i-1}}(\widetilde{B})} (\widetilde{\mathcal{F}}_{i-1}(x) \cap \widetilde{\mathcal{F}}_{\Delta X_{i-1}}(x))$. ■

Based on the discussion in Case 1, progressive learning by the update mechanism of object oriented is presented. The details of the novel cognitive mechanism are shown in Algorithm 2.

Based on the aforementioned analysis, Algorithm 2 shows the update mechanism of object-oriented CCL. Now, we can analyze the time complexity of Algorithm 2. For step 1, it will call Algorithm 1, running time complexity is $O(|U||A|(|X| + |\widetilde{B}|)))$. Thus, the time complexity of steps 2–9 is $O(|U||A||\Delta U|(|X| + |\widetilde{B}|))$. Thus, running time complexity of Algorithm 2 takes $O(|U||A||\Delta U|(|X| + |\widetilde{B}|))$.

Case 2: The update mechanism of attribute-oriented CCL.

Proposition 6: Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{H})$ be a cognitive system, $\Delta X_{i-1} = \emptyset$ and $\Delta\widetilde{B}_{i-1} \neq \emptyset$, then the following statements hold:

1) $$(\mathcal{H}_i\widetilde{\mathcal{F}}_i(X), \widetilde{\mathcal{F}}_i(X)) = \left(\bigcap_{a\in\widetilde{\mathcal{F}}_{i-1}(X)\cup\widetilde{\mathcal{F}}_{\Delta\widetilde{B}_{i-1}}(X)} (\mathcal{H}_{i-1}(a) \cap \mathcal{H}_{\Delta B_{i-1}}), \widetilde{\mathcal{F}}_{i-1}(X) \cup \widetilde{\mathcal{F}}_{\Delta\widetilde{B}_{i-1}}(X)\right);$$

2) $$(\mathcal{H}_i(\widetilde{B}), \widetilde{\mathcal{F}}_i\mathcal{H}_i(\widetilde{B})) = \left(\mathcal{H}_{i-1}(\widetilde{B}) \cap \mathcal{H}_{\Delta_{\widetilde{B}_{i-1}}}(\widetilde{B}), \bigcap_{x\in(\mathcal{H}_{i-1}(\widetilde{B})\cap\mathcal{H}_{\Delta_{\widetilde{B}_{i-1}}}(\widetilde{B}))} (\widetilde{\mathcal{F}}_{i-1}(x) \cup \widetilde{\mathcal{F}}_{\Delta\widetilde{B}_{i-1}}(x))\right).$$

Proof: We can prove it in a manner to Proposition 5. ■

Combining Definition 3, and Propositions 4–6, we know that the fuzzy granule concept in F-TCCL can be learned from the object-oriented (i.e., Case 1) and attribute-oriented CCL process (i.e., Case 2). Furthermore, we can learn the fuzzy granule concept by decomposing the learning process into object-oriented and attribute-oriented CCL processes when $\Delta X_{i-1} \neq \emptyset \& \Delta\widetilde{B}_{i-1} \neq \emptyset$. Thus, this article only discusses the situation under object-oriented and attribute-oriented CCL methods.

Based on the discussion in Case 2, progressive learning by the update mechanism of attribute-oriented is presented, and the details of the novel cognitive mechanism as shown in Algorithm 3.

According to the aforementioned discussion, Algorithm 3 gives a novel update mechanism of attribute-oriented CCL. Now, we can analyze the time complexity of Algorithm 3. For step 1, it will call Algorithm 1, running time complexity is $O(|U||A|(|X| + |\widetilde{B}|)))$. Hence, the time complexity of steps 2–9 is $O(|U||A||\Delta A|(|X| + |\widetilde{B}|))$. Thus, running time complexity of Algorithm 2 takes $O(|U||A||\Delta A|(|X| + |\widetilde{B}|))$.

V. Experiments

In this section, we validate the effectiveness of our method for the performance of CCL based on the fuzzy formal context. Specifically, we compare TCCL with other CCL models, such as TCL method [19], [25], GCCL method [20], and FCLM method [28] to compare and analyze. The experimental computing program on a personal computer and its specific configuration is OS: Microsoft WIN10; Processor: Intel(R) Core(TM) i7-10750H CPU at 2.60 GHz 2.59 GHz; Memory: 32 GB; Programming language: Python.

TABLE I
DATASET INFORMATION

No.s	Dataset	Sample	Attribute	Class
1	Iris	150	4	3
2	Banknote	1372	4	2
3	Yeast	1484	8	10
4	Winequality-Red	1599	11	11
5	Htru2	17898	8	2
6	Magic	19020	10	2
7	Hill-valley	606	101	2
8	Communities and Crime	1994	128	15
9	Communities and Crime Unnormalized	2215	147	22
10	Arrhythmia	452	279	10
11	Secom	1567	591	7
12	Isolet	7797	617	26

To extensively validate the performance of the various algorithms, we randomly selected 12 datasets with different scales from UCI Repository[1] and carried out numerical experiments under different information granule clues. The detailed information about the dataset is shown in Table I. To ensure the fuzziness of datasets before testing fuzzy-based concept-cognitive methods, we divide each data by the maximal value of its corresponding attribute. For convenience, the 12 fuzzy datasets are denoted as datasets 1–12. Meanwhile, according to the size of the object and attribute, we divide the datasets into two groups: the sample with relatively large, i.e., datasets 1–6, and the attribute with relatively large, i.e., datasets 7–12. For our purpose, in this article, the first group is used to validate the update mechanism of object-oriented concept learning, and the second group is selected to validate the update mechanism of attribute-oriented concept learning.

A. Evaluating the Performance of Concept Learning

To demonstrate the outcome of the approaches and make comparisons for static learning, we randomly take 10%, 30%, 60%, and 80% objects in each dataset as the initial X_0. We also take nine different membership functions on the fuzzy attributes as the initial $\widetilde{B}_0$. Thus, the initial information granules are $4 * 9$ pairs in each dataset. The nine membership functions considered in this paper are small, middle, and large fuzzy membership functions of Gaussian, Cauchy, and Γ membership functions. Symbols SG, MG, LG, SC, MC, LC, SΓ, MΓ, and LΓ, denote the nine membership functions, where the function setting method refers to the paper [25].

1) Number of Fuzzy Concept: This part mainly demonstrates the performance of fuzzy concepts generated by the F-TCCL theory in our article. Note that the number of fuzzy concepts generated by the method in this article is the same as that of other CCL methods, indicating our method's effectiveness. Next, the number of resulting fuzzy granule concepts of datasets are shown in Table II and Fig. 3, respectively. As seen from the resulting number of fuzzy granule concepts in Table II, when given a dataset for an arbitrary information granule, we can obtain the following conclusions.

1) The number of fuzzy granule concepts is no more than 6 through the two-way learning system. This also verifies our conclusion in Proposition 2.

[1][Online]. Available: https://www.uci.edu/

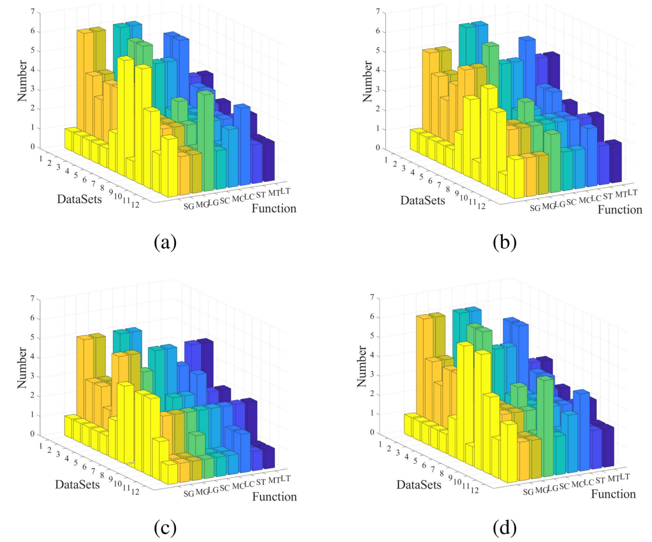

Fig. 3. Number of fuzzy concepts under different clue ratios. (a) 10%. (b) 30%. (c) 60%. (d) 80%.

2) We can learn concepts for different fuzzy membership functions through our method. In addition, compared with the TCL method in formal context [19], our method is greatly influenced by the membership function distribution of the attributes in the initial information under the fuzzy environment. In contrast, the sample size of the initial information has little influence. That is to say that the two-way learning method in a formal context and fuzzy formal context of the learning mode and cognitive mechanism is different. Thus, exploring two-way learning methods under the fuzzy formal context is necessary.

2) Learning Time: This part mainly verifies the learning time(s) in the F-TCCL system. Meanwhile, to reduce the randomness of the experiment, we still ran ten times on each dataset to obtain the average results. In the aforementioned discussion, we have verified the performance of concept learning. This part will mainly discuss the performance of the F-TCCL method in terms of learning time compared with the TCL method for static data. Next, the learning time of the resulting fuzzy concepts is shown in Table III, where the results of the algorithm with less time are displayed in bold. As seen from the comparison of learning time for three methods (i.e., F-TCCL, TCL, GCCL) in Figs. 4 and 5, when given a dataset for an arbitrary information granule, we can obtain the following conclusions.

1) The method proposed in this article saves more time than other methods to ensure the concept learning ability.
2) The learning time of the three concept learning methods in each fuzzy dataset is independent of the selected fuzzy membership function.
3) Compared with the granulation mechanism, the two-way learning method (including TCL and F-TCCL) is less affected by the sample size. However, if the granulation method is used for concept learning, the corresponding learning time will increase with the sample size. It also

TABLE II
NUMBER OF FUZZY CONCEPT

Group 1											Group 2										
No.s	Ratio	SG	MG	LG	SC	LC	LC	ST	MT	LT	No.s	Ratio	SG	MG	LG	SC	LC	LC	ST	MT	LT
1	0.1	2	2	4	3	2	5	2	2	3	7	0.1	1	1	2	1	1	2	1	1	2
	0.3	2	2	3	2	2	3	2	2	2		0.3	1	1	2	1	1	2	1	1	2
	0.6	1	1	2	1	1	1	1	1	1		0.6	1	1	1	1	1	2	1	1	2
	0.8	1	1	1	1	2	2	1	1	1		0.8	1	1	1	1	1	2	1	1	1
2	0.1	2	2	2	2	2	2	2	2	2	8	0.1	4	4	6	5	5	6	4	4	1
	0.3	1	1	1	1	1	1	1	1	1		0.3	5	5	6	5	5	6	5	5	1
	0.6	1	1	2	1	1	2	1	1	2		0.6	5	5	4	5	5	4	5	5	1
	0.8	1	1	1	1	1	1	1	1	1		0.8	5	5	3	5	5	3	5	5	1
3	0.1	3	3	3	3	3	3	3	3	4	9	0.1	2	2	6	2	2	6	2	4	1
	0.3	3	3	3	3	3	3	3	3	4		0.3	2	2	3	2	2	3	2	4	1
	0.6	3	3	3	3	3	3	3	3	4		0.6	2	2	3	2	2	3	2	2	1
	0.8	3	3	3	3	3	3	3	3	3		0.8	2	2	3	2	2	3	2	2	1
4	0.1	2	2	3	3	3	4	3	3	6	10	0.1	3	3	1	3	3	1	3	3	1
	0.3	2	2	3	3	3	4	3	3	5		0.3	3	3	1	3	3	1	3	3	1
	0.6	2	2	2	2	2	2	2	2	4		0.6	3	3	1	3	3	1	3	3	1
	0.8	1	1	1	1	1	1	1	1	1		0.8	2	2	1	2	2	1	2	2	1
5	0.1	2	2	4	3	3	1	3	3	1	11	0.1	1	1	1	1	1	1	4	4	1
	0.3	2	2	4	2	2	1	2	3	1		0.3	1	1	1	1	1	1	4	4	1
	0.6	1	1	2	1	1	0	1	2	0		0.6	1	1	1	1	1	1	3	3	1
	0.8	1	1	1	1	1	0	1	2	0		0.8	1	1	1	1	1	1	3	3	1
6	0.1	3	3	4	3	3	2	3	3	6	12	0.1	1	1	1	6	6	1	6	6	1
	0.3	3	3	4	3	3	2	3	3	4		0.3	1	1	1	6	6	1	5	5	1
	0.6	3	3	4	3	3	2	3	3	4		0.6	1	1	1	5	5	1	5	5	1
	0.8	3	3	4	3	3	2	3	3	4		0.8	1	1	1	5	5	1	5	5	1

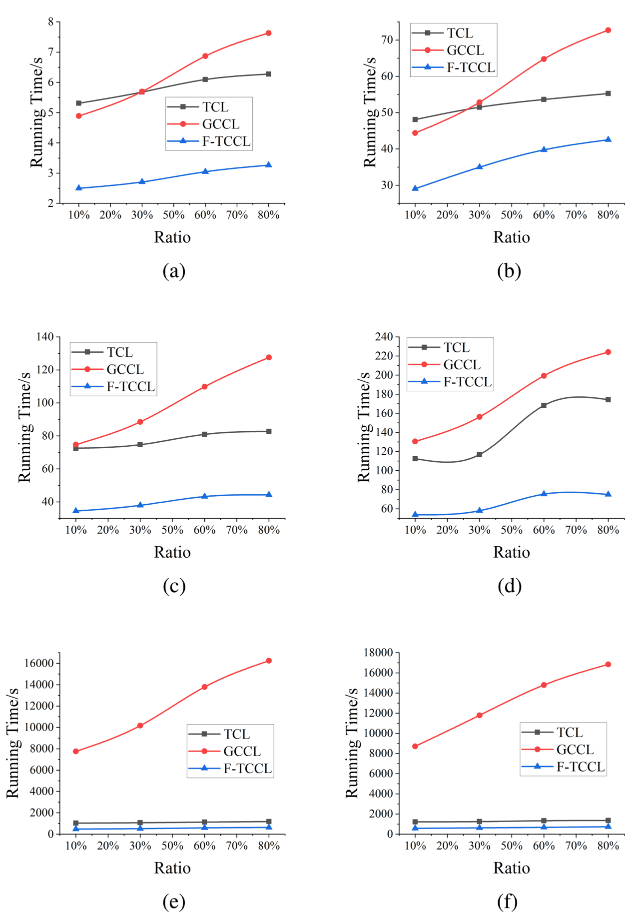

Fig. 4. Comparison of learning time on Group 1. (a) Iris. (b) Banknote. (c) Yeast. (d) Winequality-red. (e) Htru2. (f) Magic.

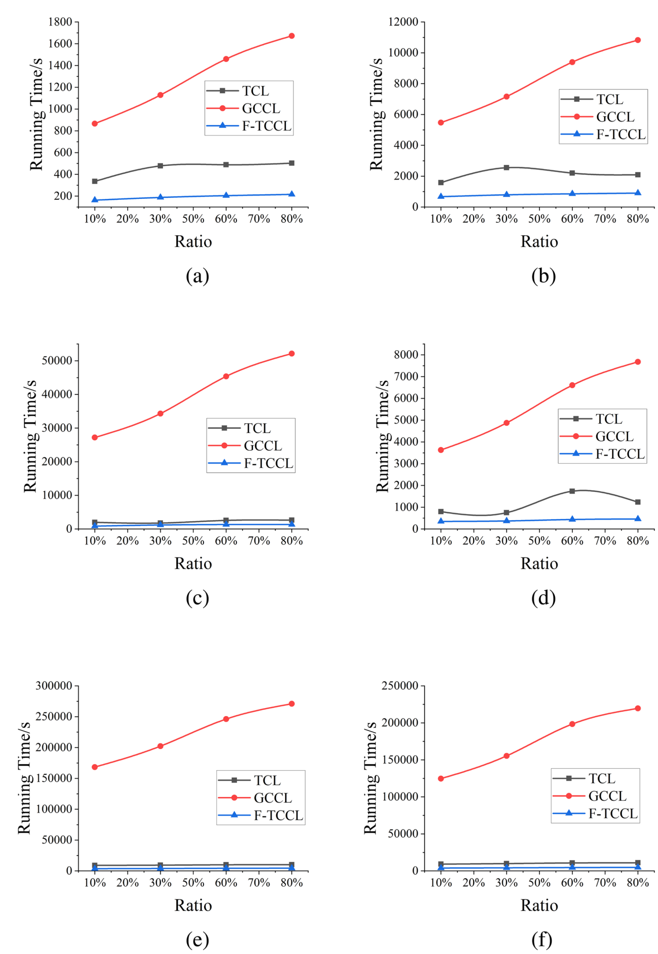

Fig. 5. Comparison of learning time on Group 2. (a) Hill-valley. (b) Communities and crime. (c) Communities and crime unnormalized. (d) Arrhythmia. (e) Secom. (f) Isoley.

TABLE III
LEARNING TIME OF FUZZY CONCEPT

			Group 1										Group 2								
No.s	Ratio	Method	SG	MG	LG	SC	MC	LC	ST	MT	LT	No.s	SG	MG	LG	SC	MC	LC	ST	MT	LT
1	0.1	TCL	5.32	5.31	6.67	5.18	5.24	5.34	5.30	5.27	5.13	7	335.62	335.39	454.14	334.52	329.75	463.64	334.67	337.07	458.57
		GCCL	4.99	4.89	8.15	4.96	4.84	5.13	4.90	4.84	4.46		872.68	866.21	912.55	858.05	854.26	932.13	859.93	856.00	890.46
		F-TCCL	**2.51**	**2.50**	**3.43**	**2.53**	**2.50**	**2.62**	**2.56**	**2.58**	**2.49**		**165.86**	**162.88**	**179.66**	**163.94**	**163.59**	**189.41**	**165.45**	**165.39**	**176.86**
	0.3	TCL	5.66	5.68	7.10	5.62	5.58	5.82	6.06	5.76	5.69		472.43	477.91	465.82	477.83	473.30	470.03	480.22	488.32	472.11
		GCCL	5.73	5.70	9.57	5.57	5.64	5.93	5.65	5.63	5.55		1122.07	1128.73	1161.62	1102.78	1122.33	1147.72	1117.27	1110.96	1113.58
		F-TCCL	**2.77**	**2.71**	**3.74**	**2.74**	**2.73**	**2.80**	**2.73**	**2.72**	**2.74**		**190.35**	**188.19**	**184.93**	**188.66**	**187.12**	**185.84**	**193.15**	**189.98**	**181.75**
	0.6	TCL	6.07	6.10	10.15	6.01	5.98	6.14	6.12	6.02	5.98		494.52	488.22	479.70	499.45	488.36	481.10	784.47	490.73	484.64
		GCCL	6.97	6.87	11.14	6.83	6.85	7.16	6.88	6.86	6.87		1482.60	1459.80	1451.50	1435.80	1447.62	1482.48	1439.77	1460.04	1444.14
		F-TCCL	**3.07**	**3.05**	**4.33**	**3.09**	**3.07**	**3.22**	**3.12**	**3.09**	**3.12**		**205.87**	**204.27**	**201.26**	**205.49**	**220.32**	**200.80**	**208.01**	**203.02**	**198.80**
	0.8	TCL	6.35	6.27	10.52	6.07	6.17	8.51	6.34	6.18	4.74		502.73	502.75	490.24	499.94	499.93	494.29	647.96	502.27	878.23
		GCCL	7.78	7.63	11.92	7.56	7.70	8.32	7.69	7.74	8.01		1710.15	1672.64	1680.71	1662.47	1694.48	1715.27	1657.41	1673.68	1682.10
		F-TCCL	**3.28**	**3.26**	**4.55**	**3.28**	**3.31**	**3.69**	**3.31**	**3.32**	**3.63**		**214.63**	**216.05**	**211.17**	**215.67**	**216.53**	**214.77**	**220.75**	**215.68**	**209.22**
2	0.1	TCL	48.63	48.09	71.76	62.33	67.47	70.26	67.45	67.36	65.37	8	1372.39	1580.14	2104.57	1310.34	1652.38	1844.41	1331.51	1357.04	2538.05
		GCCL	45.20	44.40	86.21	62.37	61.16	88.64	63.29	65.58	66.64		5467.69	5478.63	5742.93	5172.71	5277.61	5876.59	5779.66	5439.44	5307.76
		F-TCCL	**21.90**	**29.03**	**35.09**	**29.29**	**28.89**	**37.95**	**28.85**	**29.75**	**29.56**		**678.73**	**669.59**	**763.76**	**669.82**	**654.65**	**747.34**	**760.78**	**675.80**	**664.99**
	0.3	TCL	51.94	51.48	73.06	71.26	70.56	75.71	68.05	71.44	73.72		2049.14	2551.19	2334.84	1905.35	2363.18	1934.49	1906.62	1969.70	2665.05
		GCCL	53.20	52.85	93.51	73.46	77.24	93.41	74.81	71.91	82.00		7181.15	7161.36	7453.31	6551.43	6916.91	7445.05	7499.43	7168.41	6917.53
		F-TCCL	**23.95**	**34.98**	**37.63**	**34.76**	**30.76**	**37.11**	**31.59**	**32.03**	**35.53**		**797.22**	**792.44**	**816.72**	**789.64**	**770.03**	**783.20**	**808.32**	**779.72**	**778.02**
	0.6	TCL	53.91	53.62	105.37	71.43	73.70	111.07	76.09	76.18	100.22		2500.62	2199.46	2348.04	2006.37	2324.66	2552.85	1983.45	2035.67	2850.25
		GCCL	65.09	64.77	117.36	88.50	93.26	116.82	90.41	91.17	102.22		9368.92	9395.15	9546.01	8566.18	9237.03	9610.36	9731.84	9457.38	9261.96
		F-TCCL	**27.03**	**39.75**	**43.08**	**36.63**	**38.25**	**45.00**	**38.24**	**35.88**	**44.15**		**860.24**	**854.70**	**880.48**	**854.24**	**834.59**	**846.36**	**878.71**	**827.36**	**833.61**
	0.8	TCL	56.42	55.26	105.66	75.39	72.43	108.43	71.37	81.23	106.94		2698.92	2091.77	2648.93	2081.46	2144.52	2297.94	2023.43	2101.14	2890.95
		GCCL	73.93	72.72	128.73	103.23	108.14	129.73	99.20	106.53	123.66		10712.09	10824.74	10969.55	10113.96	10575.65	10719.89	11154.33	10833.10	10517.20
		F-TCCL	**30.54**	**42.53**	**47.44**	**43.33**	**43.86**	**48.88**	**43.03**	**41.60**	**46.21**		**910.17**	**904.83**	**885.42**	**905.67**	**873.20**	**886.00**	**911.34**	**882.91**	**885.64**
3	0.1	TCL	97.10	72.52	71.59	71.92	72.38	71.81	71.95	71.81	99.54	9	1722.60	1997.32	2417.29	1744.06	2718.15	2383.69	1682.41	1700.47	3498.30
		GCCL	74.68	74.69	76.69	74.76	73.95	74.73	75.14	74.76	86.21		27197.35	27203.49	27926.83	26796.75	28842.54	26988.73	26432.74	26050.01	78923.44
		F-TCCL	**33.33**	**34.44**	**33.40**	**33.51**	**33.04**	**33.79**	**33.48**	**33.96**	**38.25**		**850.52**	**862.57**	**960.47**	**862.13**	**938.19**	**943.63**	**837.81**	**866.77**	**1441.81**
	0.3	TCL	104.10	74.68	74.39	74.10	74.69	74.06	74.58	74.02	101.56		1781.41	1775.51	2665.48	2106.78	2488.46	2454.70	1787.82	1747.11	4176.94
		GCCL	88.48	88.49	88.99	88.74	87.98	88.63	89.25	89.30	99.08		32683.04	34312.81	35296.14	34386.28	35342.65	34358.23	33603.88	32947.94	88235.25
		F-TCCL	**36.62**	**37.90**	**35.89**	**36.95**	**35.97**	**36.41**	**36.75**	**37.44**	**41.15**		**909.04**	**1191.09**	**1016.95**	**911.41**	**1004.67**	**1008.55**	**890.70**	**897.95**	**1450.03**
	0.6	TCL	115.58	80.92	80.60	80.41	80.54	79.89	80.37	80.21	107.70		3157.08	2569.29	3055.04	2956.18	2636.51	2570.83	2571.93	2558.31	3977.44
		GCCL	109.02	109.79	111.10	108.78	108.40	109.05	109.97	108.13	118.27		42595.34	45358.98	46126.83	45401.98	45782.35	45088.65	44470.50	43922.79	98888.78
		F-TCCL	**42.27**	**43.24**	**41.95**	**42.40**	**41.70**	**42.08**	**42.52**	**43.11**	**47.05**		**1085.62**	**1343.54**	**1111.24**	**1085.68**	**1078.98**	**1073.78**	**1065.47**	**1094.53**	**1513.61**
	0.8	TCL	119.72	82.72	82.05	81.84	82.23	81.42	82.11	81.92	111.29		3156.53	2624.18	2947.58	3287.72	2680.93	2646.24	2593.58	2615.39	3712.14
		GCCL	126.17	127.58	124.78	122.47	121.73	125.62	123.91	121.85	133.54		50317.71	52148.98	53208.97	53420.07	53058.87	51924.23	51104.26	51390.16	106095.84
		F-TCCL	**43.83**	**44.26**	**43.34**	**44.08**	**43.00**	**43.10**	**43.95**	**43.98**	**48.40**		**1172.42**	**1344.62**	**1197.53**	**1150.70**	**1139.38**	**1123.45**	**1114.22**	**1137.76**	**1567.25**
4	0.1	TCL	111.77	112.57	158.00	112.39	112.08	155.45	113.07	113.29	205.94	8	708.99	795.78	1264.48	813.55	712.88	1288.44	706.77	701.61	1319.73
		GCCL	129.53	130.55	140.51	130.48	129.56	147.92	129.84	129.66	291.47		3626.16	3628.96	10690.35	3522.23	3495.88	10523.09	3607.87	3613.53	10540.73
		F-TCCL	**53.44**	**53.79**	**59.13**	**53.16**	**52.54**	**59.36**	**53.05**	**53.20**	**83.09**		**346.81**	**349.68**	**523.01**	**348.95**	**344.13**	**515.07**	**345.94**	**347.84**	**544.10**
	0.3	TCL	114.82	116.78	160.59	115.48	115.68	159.16	116.06	116.27	212.52		756.69	750.81	1775.07	742.73	743.32	1591.10	755.19	1053.81	1746.29
		GCCL	154.34	156.26	164.49	152.86	154.23	170.18	152.74	153.11	315.36		4784.15	4875.47	12820.65	4780.93	4720.24	12682.24	4758.49	4778.22	12710.83
		F-TCCL	**57.10**	**58.01**	**63.10**	**57.60**	**57.08**	**63.47**	**57.02**	**57.12**	**88.80**		**370.70**	**369.50**	**549.55**	**364.70**	**361.96**	**548.68**	**365.88**	**364.87**	**578.33**
	0.6	TCL	166.61	168.30	170.70	166.15	166.64	170.92	167.80	168.07	219.64		1088.47	1736.08	1650.19	1091.28	1086.12	1892.81	1438.63	1288.39	2037.51
		GCCL	202.46	199.32	199.50	200.28	197.91	209.82	202.03	201.17	357.48		6487.87	6604.92	16421.08	6510.33	6476.20	14607.83	6490.60	6499.11	14641.74
		F-TCCL	**71.13**	**75.37**	**70.98**	**70.73**	**70.85**	**71.73**	**70.80**	**70.42**	**94.45**		**432.99**	**438.01**	**587.13**	**432.47**	**437.28**	**581.97**	**431.39**	**431.68**	**606.41**
	0.8	TCL	175.83	174.28	170.59	172.12	169.51	170.92	170.60	171.67	223.81		1118.49	1233.19	1804.60	1120.87	1133.60	1672.01	1206.06	1358.83	1766.87
		GCCL	226.92	224.13	224.29	224.22	223.35	244.99	223.11	222.09	378.60		7667.65	7678.23	15859.52	7701.53	7717.01	16022.69	7725.38	7752.52	16001.25
		F-TCCL	**74.66**	**75.02**	**73.91**	**73.54**	**73.54**	**75.49**	**74.23**	**73.95**	**98.97**		**454.15**	**459.75**	**609.93**	**457.18**	**450.88**	**605.37**	**450.26**	**457.79**	**629.58**
5	0.1	TCL	1036.35	1040.56	1508.11	1047.11	1048.30	2369.88	1065.01	1128.50	2159.63	11	8576.99	9004.94	8982.61	8895.11	8879.97	9026.34	8902.29	9194.51	9900.17
		GCCL	7780.55	7755.01	8973.42	7777.52	7856.31	24935.13	7820.87	7826.45	26122.97		168362.03	168328.26	30320.02	61696.27	63324.05	29972.71	65137.55	65494.74	29955.05
		F-TCCL	**484.72**	**481.37**	**571.42**	**494.14**	**514.65**	**842.67**	**489.27**	**492.48**	**882.70**		**3628.58**	**3610.27**	**3695.41**	**3795.81**	**3768.77**	**3843.84**	**3774.76**	**3923.71**	**4087.85**
	0.3	TCL	1085.34	1073.97	1565.70	1084.69	1090.69	6985.71	1079.86	1907.19	2210.99		9307.02	9359.40	9446.48	9600.82	9624.89	9780.80	9652.14	9591.86	10359.28
		GCCL	9755.61	10170.79	11265.71	10243.65	10334.61	27847.79	10135.40	10391.91	28965.31		168362.03	202291.16	70929.46	80704.22	82499.48	68607.54	84047.73	84780.88	68283.78
		F-TCCL	**522.53**	**519.34**	**624.57**	**528.50**	**519.44**	**900.57**	**524.77**	**576.36**	**921.72**		**3891.94**	**3895.28**	**3981.27**	**4050.68**	**4032.41**	**4109.50**	**4046.62**	**4367.92**	**4365.57**
	0.6	TCL	1144.85	1130.93	2033.49	1631.42	1155.67	2524.48	1161.72	1828.16	2357.06		9984.14	9961.43	10122.52	10223.05	10264.73	10429.03	10295.62	10251.09	11041.64
		GCCL	12969.73	13791.95	14697.19	13861.72	14094.97	32048.48	13826.02	14011.81	32596.61		242905.05	246164.72	106671.90	107873.03	110787.71	106868.09	112735.30	112878.09	107363.38
		F-TCCL	**592.88**	**593.49**	**665.91**	**600.60**	**590.26**	**938.24**	**594.77**	**648.17**	**985.69**		**4206.42**	**4234.08**	**4318.48**	**4371.65**	**4367.32**	**4395.55**	**4330.39**	**4357.75**	**4898.94**
	0.8	TCL	1187.48	1185.59	1879.96	1531.01	1230.66	2352.18	1214.54	1699.89	2370.71		10145.68	10121.13	10306.66	10469.12	10469.00	10637.09	10494.72	10471.58	11230.11
		GCCL	15161.24	16249.24	16905.32	16372.45	16467.81	34651.66	16281.29	16546.73	34552.11		266481.45	270939.98	126776.45	127198.079	129173.01	127794.15	131391.61	131146.79	128385.06
		F-TCCL	**633.79**	**629.75**	**824.56**	**642.55**	**630.59**	**991.02**	**653.14**	**692.45**	**1041.94**		**4420.49**	**4407.56**	**4470.09**	**4523.22**	**4502.12**	**4578.53**	**4488.14**	**4498.26**	**5147.74**
6	0.1	TCL	1217.96	1222.63	2368.36	1225.68	1371.56	2382.83	1210.55	1610.25	1817.63	12	9609.19	9157.00	9460.88	9249.18	8848.51	9611.23	9274.72	9581.75	10205.42
		GCCL	9503.02	8712.75	27297.43	7604.56	7928.02	26744.31	8306.64	8022.46	12472.28		125906.06	124702.83	27648.25	77425.98	77202.59	27421.12	97246.09	98207.60	27584.10
		F-TCCL	**594.36**	**578.13**	**969.87**	**566.69**	**566.54**	**984.89**	**569.60**	**566.59**	**703.98**		**4058.14**	**3915.05**	**3956.95**	**3890.35**	**3894.59**	**3949.75**	**4200.99**	**3911.49**	**4218.11**
	0.3	TCL	1262.99	1245.82	2819.00	1250.73	1518.21	2429.35	1690.48	1249.02	1876.74		10094.96	9900.43	9974.79	9530.76	9557.05	9989.12	9826.52	9856.33	10896.40
		GCCL	12570.20	11792.93	29253.42	10490.65	10139.29	31711.80	11245.20	11206.37	14685.76		155682.29	155443.76	68924.05	99784.68	100429.56	68639.39	123426.16	124690.08	69295.86
		F-TCCL	**651.96**	**627.85**	**1007.43**	**633.97**	**618.60**	**1010.51**	**628.15**	**631.14**	**760.93**		**4275.33**	**4178.27**	**4205.45**	**4115.84**	**4209.11**	**4441.07**	**4178.45**	**4154.06**	**4482.10**
	0.6	TCL	1353.50	1332.51	3237.72	1343.82	1567.23	2521.63	1559.09	1988.36	2328.93		10881.43	10758.06	10872.14	10380.67	10323.71	10912.58	10674.46	10717.51	11445.03
		GCCL	17022.54	14796.31	33585.90	14269.78	14458.18	35353.34	14883.44	14820.50	17942.72		199670.73	198413.25	119757.83	134020.37	134320.55	119602.99	161613.57	163555.94	121201.31
		F-TCCL	**693.13**	**676.03**	**1066.39**	**681.44**	**675.93**	**1091.44**	**682.63**	**674.53**	**822.36**		**4692.37**	**4541.05**	**4578.77**	**4459.86**	**4673.63**	**4579.21**	**4580.45**	**4527.39**	**4857.02**
	0.8	TCL	1797.51	1358.38	2889.47	1369.57	1477.26	2593.29	1577.86	1699.45	2501.86		11093.52	10979.82	11079.82	10557.22	10502.15	11137.49	10892.26	10937.28	11685.31
		GCCL	19243.37	16841.74	35770.05	16513.49	17287.02	37189.38	18616.46	17939.42	20072.17		220935.39	219559.96	142097.26	154487.86	154375.26	143408.40	183861.23	184360.74	143685.46
		F-TCCL	**758.84**	**740.90**	**1128.00**	**743.11**	**742.89**	**1134.46**	**735.92**	**741.25**	**855.25**		**4741.03**	**4706.16**	**4752.73**	**4517.99**	**4493.38**	**4756.71**	**4674.13**	**4713.54**	**5016.31**

shows that two-way learning, cognitive concepts from given clues, is essential for studying CCL.

B. Evaluating the Performance of Dynamic Concept Learning

Note that F-TCCL can learn new fuzzy granule concepts from arbitrary information granules, and we also validate the concept learning performance in the last subsection. Next, to extensively validate the performance of F-TCCL for dynamic data (i.e., P-FTCCL). According to the discussion in the last subsection, we know that the learning time of the two-way learning system is less affected by the sample size and fuzzy membership function. Thus, we only select 30% objects as initial X_0 and MG as membership function in this subsection without loss of generality. In addition, we take the first 50% dataset samples in group 1 as initialization samples and 50% dataset attributes in group 2 as initialization attributes, and the remaining samples and attributes are divided into ten chunks (i.e., chunk 1, chunk 2, $\cdots$, chunk 10), each of which accounted for 5% (i.e., 55%, 60%, ..., 100%). Then, the learning time of the five methods is shown in Tables V and VI, and we can obtain the following conclusions.

1) Number of Fuzzy Concept: This part mainly verifies the fuzzy concepts generated performance of the P-FTCCL method for dynamic concept learning. The P-FTCCL and F-TCCL methods are the same regarding the number of concept generations, indicating that the method in this article is effective for learning concepts in a dynamic environment. Next, the number of

TABLE IV
NUMBER OF FUZZY CONCEPT BY DYNAMIC LEARNING

Group1											Group 2										
No.s	Chunk1	Chunk2	Chunk3	Chunk4	Chunk5	Chunk6	Chunk7	Chunk8	Chunk9	Chunk10	No.s	Chunk1	Chunk2	Chunk3	Chunk4	Chunk5	Chunk6	Chunk7	Chunk8	Chunk9	Chunk10
1	2	2	2	2	2	2	2	2	2	1	7	3	3	3	3	3	3	3	3	3	2
2	5	5	5	5	5	5	5	5	5	5	8	5	5	5	5	5	5	5	5	5	5
3	4	4	4	5	5	5	5	5	5	5	9	3	3	3	5	5	5	5	5	5	5
4	5	5	5	5	5	5	5	5	5	4	10	6	6	6	6	6	6	6	6	6	6
5	6	6	6	6	6	6	6	6	6	6	11	6	6	6	6	6	6	6	6	6	5
6	6	6	6	6	6	6	6	6	6	6	12	6	6	6	6	6	6	6	6	6	6

TABLE V
LEARNING TIME OF FUZZY CONCEPT FOR DYNAMIC UPDATE ON GROUP 1

No.s	Method	Chunk 1	Chunk 2	Chunk 3	Chunk 4	Chunk 5	Chunk 6	Chunk 7	Chunk 8	Chunk 9	Chunk 10
1	TCL	5.43	6.19	6.35	7.07	7.212	7.714	8.045	9.11	8.96	9.55
	FCLM	6.14	6.33	6.67	7.54	7.544	8.814	8.673	9.08	9.27	10.44
	GCCL	3.09	3.27	3.54	3.87	4.089	4.371	4.999	4.87	5.07	6.69
	F-TCCL	2.68	2.47	2.71	2.82	3.186	3.224	3.362	3.58	3.74	4.26
	P-FTCCL	**1.84**	**1.91**	**2.09**	**2.63**	**2.46**	**2.57**	**2.83**	**2.90**	**2.96**	**3.27**
2	TCL	43.38	47.84	50.8	53.95	58.05	61.29	66.06	68.75	71.92	76.13
	FCLM	41.25	44.95	48.38	52.32	55.66	58.15	61.57	65.08	68.96	72.25
	GCCL	23.46	25.44	27.38	28.77	31.13	33.25	34.69	37.76	39.28	41.44
	F-TCCL	16.91	17.64	19.22	20.30	21.54	23.58	24.73	26.26	27.18	29.67
	P-FTCCL	**12.81**	**14.09**	**15.33**	**16.20**	**17.67**	**18.76**	**19.53**	**20.56**	**24.87**	**23.35**
3	TCL	74.27	80.07	86.22	93.49	98.68	105.67	126.94	119.08	126.75	131.85
	FCLM	88.82	96.62	105.29	114.51	122.56	136.54	137.83	148.11	157.36	165.26
	GCCL	50.99	54.69	59.85	63.34	69.24	73.06	77.22	98.34	86.18	91.25
	F-TCCL	31.43	33.41	36.13	40.35	50.34	42.20	44.71	47.25	50.78	52.38
	P-FTCCL	**22.09**	**24.11**	**26.96**	**28.40**	**30.03**	**32.21**	**34.65**	**36.63**	**38.14**	**40.26**
4	TCL	119.11	126.37	137.21	149.7	173.62	172.20	185.2	190.07	203.09	211.93
	FCLM	164.62	180.68	195.45	212.79	229.97	243.34	261.77	278.70	293.16	308.83
	GCCL	99.63	107.67	117.96	126.96	152.03	144.52	154.45	164.18	164.18	183.84
	F-TCCL	48.18	50.97	55.64	59.49	63.67	69.00	71.99	77.15	80.31	84.98
	P-FTCCL	**35.11**	**37.83**	**41.11**	**61.65**	**47.46**	**50.58**	**54.17**	**57.32**	**60.07**	**63.49**
5	TCL	1098.53	1179.68	1311.73	1376.93	1499.21	1580.35	1704.52	1800.58	1869.77	2065.02
	FCLM	2197.95	2445.86	2765.39	3029.07	3391.23	3720.41	17816.67	19944.10	21911.08	23094.16
	GCCL	941.47	1049.44	1158.11	1304.93	1396.16	1521.27	5310.18	5850.51	6350.44	6931.49
	F-TCCL	444.20	467.20	519.76	565.46	591.51	659.39	688.45	726.02	742.56	814.63
	P-FTCCL	**328.42**	**351.50**	**378.14**	**410.21**	**435.55**	**520.34**	**526.31**	**530.92**	**577.92**	**597.96**
6	TCL	919.89	997.10	1081.27	1192.12	1252.46	1405.58	1485.66	1526.81	1632.2	1705.00
	FCLM	1080.33	1191.37	1314.65	5066.30	5662.14	6737.13	7624.68	8436.38	9348.51	10498.81
	GCCL	786.99	900.19	981.15	4162.87	4682.30	5311.52	5784.49	6515.82	7198.98	7734.16
	F-TCCL	337.57	366.21	395.69	442.66	458.48	492.32	532.78	565.20	612.21	627.73
	P-FTCCL	**304.09**	**334.21**	**355.48**	**384.36**	**402.22**	**432.12**	**456.41**	**484.03**	**529.11**	**537.73**

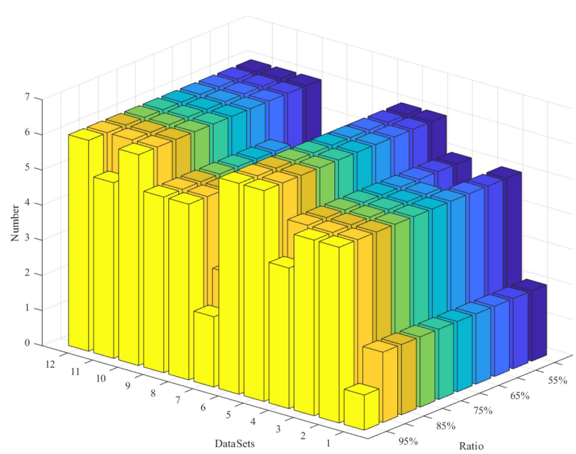

Fig. 6. Number of fuzzy concept by dynamic learning.

resulting fuzzy granule concepts of datasets is shown in Table IV and Fig. 6, respectively.

2) Dynamic Learning Time: This part mainly compares the learning time of different CCL methods (including F-TCCL, P-FTCCL, TCL, GCCL, and FCLM) for dynamic data to explain the advantages of the fuzzy-based progressive learning method in this article. Then, the dynamic learning time of the resulting fuzzy concepts is shown in Tables V and VI. In these tables, the results of the algorithm with less time are displayed in bold. The comparison of learning time for three methods (i.e., F-TCCL, TCL, and GCCL) is shown in Figs. 7 and 8. For an arbitrary information granule, we can obtain the following conclusions: compared with other CCL methods, the proposed method has better performance in learning time when facing dynamic data updates. With the increase in dataset size, our method shows good adaptability. In the case of the large samples and the large attribute datasets, our method can also consume very little time, showing that the method in this article is suitable for concept learning of Big Data.

C. Comparative Analysis Between Five CCL Mechanisms

According to the discussion in this section, we compare our method with other CCL mechanisms, including 2WL mechanism [19], TCL mechanism [25], GCCL mechanism [20], and FCLM mechanism [28]. Thus, we further make the

TABLE VI
LEARNING TIME OF FUZZY CONCEPT FOR DYNAMIC UPDATE ON GROUP 2

No.s	Method	Chunk 1	Chunk 2	Chunk 3	Chunk 4	Chunk 5	Chunk 6	Chunk 7	Chunk 8	Chunk 9	Chunk 10
	TCL	244.65	269.91	287.02	311.00	333.68	358.64	389.52	403.05	427.95	485.11
	FCLM	448.77	508.10	582.016	651.93	735.99	810.77	900.2	1010.12	1069.27	1214.71
7	GCCL	355.30	432.019	479.96	533.70	611.61	677.72	771.23	844.94	924.16	1029.99
	F-TCCL	93.52	102.97	110.35	118.28	127.11	135.72	142.49	152.08	161.65	169.84
	P-FTCCL	**54.71**	**60.13**	**65.94**	**69.90**	**77.80**	**81.78**	**86.67**	**98.66**	**95.41**	**102.30**
	TCL	1198.43	1317.45	1445.51	1566.22	1694.57	1816.13	1895.97	2011.18	2180.18	2260.38
	FCLM	5130.66	6365.49	7183.37	7761.69	8639.05	9578.28	10446.82	11405.98	12430.22	13553.37
8	GCCL	3391.86	4061.90	4610.08	5173.03	6000.90	6589.31	7284.49	8225.89	9198.87	10096.80
	F-TCCL	471.67	515.43	558.64	598.44	647.60	697.37	729.09	785.61	814.57	864.69
	P-FTCCL	**291.62**	**325.58**	**350.54**	**374.92**	**414.04**	**434.17**	**466.88**	**488.85**	**523.94**	**539.97**
	TCL	1277.81	1447.58	1541.81	1686.94	1787.38	1923.58	2038.16	2169.07	2281.12	2426.9
	FCLM	8249.98	10912.57	15084.63	16323.56	17590.17	20455.59	25248.91	30807.44	32349.32	34866.44
9	GCCL	6343.71	8428.10	11822.40	12909.38	14006.84	16676.86	20471.72	24990.27	26593.74	29034.21
	F-TCCL	495.64	548.78	580.18	632.26	667.23	714.50	760.89	813.54	850.00	896.91
	P-FTCCL	**295.25**	**324.67**	**352.63**	**379.49**	**408.00**	**437.00**	**463.62**	**485.84**	**510.03**	**555.98**
	TCL	598.05	655.22	1023.97	776.90	1285.32	915.23	1040.87	1172.16	1083.42	1456.54
	FCLM	2860.08	3345.45	3855.78	4405.05	5010.05	5626.93	6333.63	7010.40	7943.49	8729.89
10	GCCL	2534.47	3003.43	3473.90	4008.65	4602.71	5211.48	5866.68	6569.68	7294.89	8056.15
	F-TCCL	228.71	250.72	272.38	292.19	316.30	349.58	363.73	388.07	424.07	433.54
	P-FTCCL	**138.31**	**151.09**	**163.10**	**175.99**	**191.62**	**202.18**	**216.29**	**236.20**	**245.68**	**258.87**
	TCL	4950.49	5426.89	5870.96	6382.42	6860.21	7357.10	7826.87	8344.09	8621.76	9148.69
	FCLM	69562.54	81836.44	94989.79	110102.93	124279.32	141616.82	157988.65	177002.74	190305.62	209920.30
11	GCCL	59204.17	70679.45	82870.59	95862.19	109958.34	125682.34	141305.46	159768.36	173690.64	195106.36
	F-TCCL	2052.61	2245.03	2468.89	2650.14	2831.75	3023.21	3198.16	3388.73	3495.12	3677.23
	P-FTCCL	**1261.45**	**1377.22**	**1509.54**	**1610.42**	**1724.89**	**1854.31**	**1968.33**	**2089.64**	**2198.93**	**2283.79**
	TCL	5855.92	5477.15	5358.12	5779.44	6201.34	6424.82	6849.73	7333.14	7797.05	8243.70
	FCLM	50928.29	59910.46	70204.83	78970.62	90926.27	99229.68	111960.34	125674.44	138655.53	153781.15
12	GCCL	44412.64	53249.76	61888.57	71073.42	81530.48	89809.51	100973.31	112559.23	125069.81	139463.05
	F-TCCL	1777.26	1940.93	2095.67	2238.47	2407.59	2560.40	2688.66	2861.87	3043.83	3215.74
	P-FTCCL	**1076.90**	**1195.38**	**1298.00**	**1389.87**	**1491.93**	**1567.79**	**1662.38**	**1763.39**	**1901.22**	**1961.57**

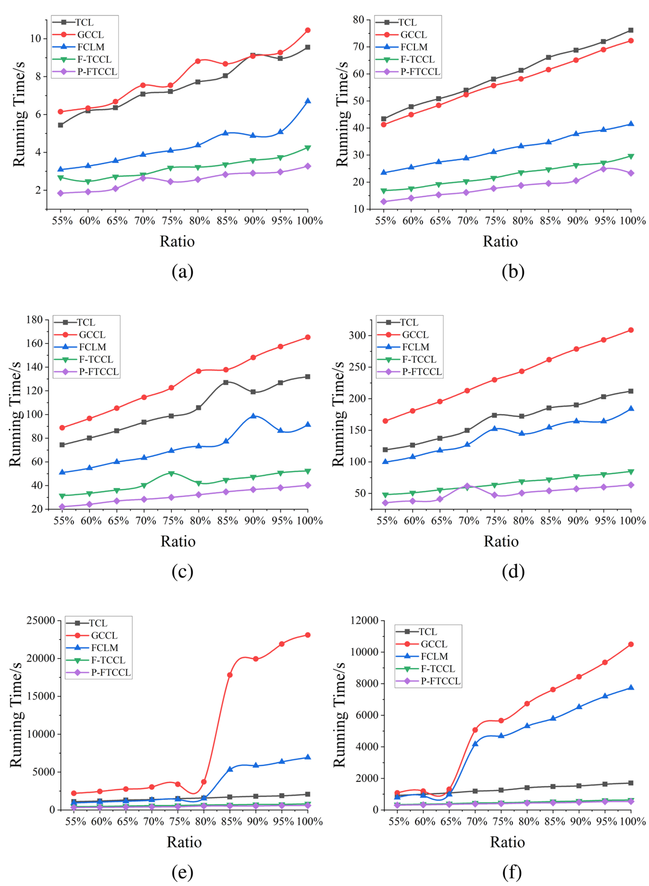

Fig. 7. Comparison of dynamic learning on Group 1. (a) Iris. (b) Banknote. (c) Yeast. (d) Winequality-red. (e) Htru2. (f) Magic.

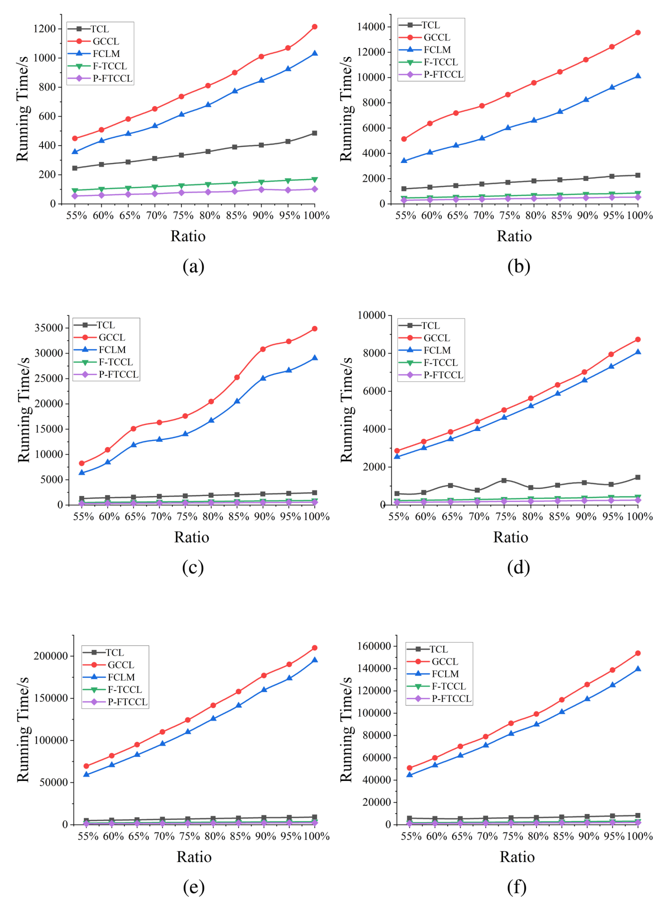

Fig. 8. Comparison of running time on Group 2. (a) Hill-valley. (b) Communities and crime. (c) Communities and crime unnormalized. (d) Arrhythmia. (e) Secom. (f) Isoley.

TABLE VII
COMPARATIVE ANALYSIS BETWEEN FIVE CCL MECHANISMS

	2WL	TCL	FCLM	GCCL	TCCL
Fuzzy environment	○	●	●	●	●
Dynamic environment	○	○	●	●	●
Learning of clues	●	●	○	○	●
Concept space	○	○	●	●	●
Running time	○	long	very long	long	short

Note: "●" denotes yes and "○" denotes no, "short" denotes short learning time for granule concept, "long" denotes long learning time, and "very long" denotes very long learning time.

comparative analysis between TCCL mechanism (including F-TCCL and P-FTCCL mechanisms) and other mechanisms.

For clarity, all the similarities and differences are summed in Table VII. As shown in Table VII, the TCL, FCLM, and GCCL mechanisms and our method have three same characteristics. The five mechanisms consider the fuzzy environment, dynamic environment, learning of clues, concept space, and running time. This can also reflect that it is essential and reasonable to study our method to process the concept learning of given clues and learning time under fuzzy and dynamic environments.

Both four CCL mechanisms utilize a pair of cognitive operators to learn the concept from the fuzzy environment, except the 2WL mechanism. However, as shown in Table VII, two main contributions and advantages are obtained in our model. First, our method introduces the two-way learning method into CCL theory to learn the concept from a given clue $(X, \widetilde{B})$, where the concept spaces of different cases (i.e.,$\widetilde{\mathcal{G}}_1, \widetilde{\mathcal{G}}_2, \widetilde{\mathcal{G}}_1 \cap \widetilde{\mathcal{G}}_2$) are also involved. Second, the learning time of clues in our model is shorter than that of other CCL mechanisms, which is more straightforward in the dynamic environment. Then, we also make the comparison and discussion between five CCL mechanisms.

VI. CONCLUSION

F-TCCL is an effective cognitive system to describe the human cognitive process, and it can achieve the decision task via an initial information $(X, \widetilde{B})$. The essence of F-TCCL is learning more from the unknown through a pair of cognitive operators (i.e., extent-intent and intent-extent) and mapping it into fuzzy granule concept space for fuzzy datasets. Progressive learning is a novel TCCL mechanism for dynamic data updating. Generally, the sufficient and necessary fuzzy granule concept is a fuzzy concept from a given clue $(X, \widetilde{B})$. That is to say that learning pseudoconcept based on the known clue is crucial, especially in decision-making problems. The current article introduces a novel CCL system for fuzzy concept cognitive and dynamic learning in a fuzzy formal context. As an extension of two-way concept learning, the TCCL theory learns concepts in two ways, including F-TCCL and P-FTCCL. Furthermore, corresponding experiments on various datasets demonstrate the effectiveness of the proposed TCCL compared with other mechanisms.

The CCL theory is an emerging tool for data analysis and knowledge discovery. The central notion of CCL is to describe the concept through a pair of concept-cognitive operators (i.e., intent learning operator and extent learning operator), and then, use the learned concept to discover the potential knowledge in data from an arbitrarily given clue. Up to now, we have witnessed a growing interest and the development of CCL from the view of granular computing and machine learning. The core themes include the concept cognition mechanism, concept learning method, cognitive system construction mechanism, complex decision optimization mechanism, etc. Our work mainly explores the cognitive representation and learning mechanism of fuzzy concepts via a novel two-way learning approach under a dynamic environment, which can be applied to many areas, such as formal concept analysis, decision making, object classification, etc.

This article studies learning a fuzzy granule concept from a given clue and dynamic learning based on a single formal context. Hence, there still exist some limitations that need to be concerned about. Such as how to learn concepts for a multisource formal context or high-dimensional formal context via our approach, especially in handling unbalanced data. Although our method can significantly improve the efficiency of concept learning and save time to a certain extent, it still cannot be learned for billions of data. Hence, how to combine quantum computing theory into the CCL method also deserves to be investigated [32]. Our future work will continue to focus on these points.

ACKNOWLEDGMENT

The authors would like to thank the Associate Editor and the reviewers for their insightful comments and suggestions.

REFERENCES

[1] Y. Y. Yao, “Tri-level thinking: Models of three-way decision,” *Int. J. Mach. Learn. Cybern.*, vol. 11, pp. 947–959, May 2020.
[2] G. A. Miller, “The magical number seven, plus or minus two: Some limits on our capacity for processing information,” *Psychol. Rev.*, vol. 63, pp. 81–97, Feb. 1956.
[3] N. Chomsky, “Three models for the description of language,” *IEEE Trans. Inf. Theory*, vol. TIT-2, no. 3, pp. 113–124, Sep. 1956.
[4] Y. X. Wang, “On concept algebra: A denotational mathematical structure for knowledge and software modeling,” *Int. J. Cogn. Inform. Natural Intell.*, vol. 2, pp. 1–19, Apr. 2008.
[5] B. M. Lake, R. Salakhutdinov, and J. B. Tenenbaum, “Human-level concept learning through probabilistic program induction,” *Science*, vol. 350, pp. 1332–1338, Dec. 2015.
[6] R. Wille, “Restructuring lattice theory: An approach based on hierarchies of concepts,” in *Ordered Sets*, I. Rival Ed., Dordrecht, The Netherlands: Springer, 1982, pp. 445–470.
[7] L. Antoni, M. E. Cornejo, J. Medina, and E. Ramírez-Poussa, “Attribute classification and reduct computation in multi-adjoint concept lattices,” *IEEE Trans. Fuzzy Syst.*, vol. 29, no. 5, pp. 1121–1132, May 2021.
[8] J. Niu, D. Chen, J. Li, and H. Wang, “Fuzzy rule based classification method for incremental rule learning,” *IEEE Trans. Fuzzy Syst.*, vol. 30, no. 9, pp. 3748–3761, Sep. 2022, doi: 10.1109/TFUZZ.2021.3128061.
[9] Y. Yao, “Interpreting concept learning in cognitive informatics and granular computing,” *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 39, no. 4, pp. 855–866, Aug. 2009.
[10] J. J. Niu, C. C. Huang, J. H. Li, and M. Fan, “Parallel computing techniques for concept-cognitive learning based on granular computing,” *Int. J. Mach. Learn. Cybern.*, vol. 9, pp. 1785–1805, Nov. 2018.
[11] L. D. Wang and X. D. Liu, “Concept analysis via rough set and AFS algebra,” *Inf. Sci.*, vol. 178, pp. 4125–4137, Nov. 2008.
[12] J. H. Li, C. C. Huang, J. J. Qi, Y. H. Qian, and W. Q. Liu, “Three-way cognitive concept learning via multi-granularity,” *Inf. Sci.*, vol. 378, pp. 244–263, Feb. 2017.
[13] K. H. Yuan, W. H. Xu, W. T. Li, and W. P. Ding, “An incremental learning mechanism for object classification based on progressive fuzzy three-way concept,” *Inf. Sci.*, vol. 584, pp. 127–147, Jan. 2022.

[14] X. Zhu, W. Pedrycz, and Z. Li, “A development of hierarchically structured granular models realized through allocation of information granularity,” *IEEE Trans. Fuzzy Syst.*, vol. 29, no. 12, pp. 3845–3858, Dec. 2021.
[15] Y. Y. Yao, “Granular computing: Basic issues and possible solutions,” in *Proc. 5th Joint Conf. Inf. Sci.*, 2000, pp. 186–189.
[16] L. A. Zadeh, “Fuzzy sets,” *Inf. Control*, vol. 8, pp. 338–353, Jun. 1965.
[17] Z. Pawlak, “Rough sets,” *Int. J. Comput. Inf. Sci.*, vol. 11, pp. 341–356, Oct. 1982.
[18] W. X. Zhang and W. H. Xu, “Cognitive model based on granular computing,” *Chin. J. Eng. Math.*, vol. 24, pp. 957–971, Jun. 2007.
[19] W. H. Xu, J. Z. Pang, and S. Q. Luo, “A novel cognitive system model and approach to transformation of information granules,” *Int. J. Approx. Reasoning*, vol. 55, pp. 853–866, Mar. 2014.
[20] J. H. Li, C. L. Mei, W. H. Xu, and Y. H. Qian, “Concept learning via granular computing: A cognitive viewpoint,” *Inf. Sci.*, vol. 298, pp. 447–467, Mar. 2015.
[21] R. Shivhare and A. K. Cherukuri, “Three-way conceptual approach for cognitive memory functionalities,” *Int. J. Mach. Learn. Cybern.*, vol. 8, pp. 21–34, Feb. 2017.
[22] Y. Shi, Y. Mi, J. Li, and W. Liu, “Concept-cognitive learning model for incremental concept learning,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 51, no. 2, pp. 809–821, Feb. 2021.
[23] Y. Shi, Y. L. Mi, J. H. Li, and W. Q. Liu, “Concurrent concept-cognitive learning model for classification,” *Inf. Sci.*, vol. 496, pp. 65–81, Sep. 2019.
[24] W. H. Xu and Y. Q. Chen, “Multi-attention concept-cognitive learning model: A perspective from conceptual clustering,” *Knowl.-Based Syst.*, vol. 252, pp. 109472, Sep. 2020.
[25] W. H. Xu and W. T. Li, “Granular computing approach to two-way learning based on formal concept analysis in fuzzy dataset,” *IEEE Trans. Cybern.*, vol. 46, no. 2, pp. 366–379, Feb. 2016.
[26] M. Hu, E. C. C. Tsang, Y. Guo, Q. Zhang, D. Chen, and W. Xu, “A novel approach to concept-cognitive learning in interval-valued formal contexts: A granular computing viewpoint,” *Int. J. Mach. Learn. Cybern.*, vol. 13, pp. 1049–1064, 2022, doi: 10.1007/s13042-021-01434-1.
[27] W. X. Zhang, J. M. Ma, and S. Q. Fan, “Variable threshold concept lattices,” *Inf. Sci.*, vol. 177, pp. 4883–4892, Nov. 2007.
[28] Y. Mi, Y. Shi, J. Li, W. Liu, and M. Yan, “Fuzzy-based concept learning method: Exploiting data with fuzzy conceptual clustering,” *IEEE Trans. Cybern.*, vol. 52, no. 1, pp. 582–593, Jan. 2022, doi: 10.1109/TCYB.2020.2980794.
[29] B. Ganter and R. Wille, *Formal Concept Analysis, Mathematical Foundations*. Berlin, Germany: Springer, 1999.
[30] L. A. Zadeh, “Fuzzy sets and information granularity,” in *Advances in Fuzzy Set Theory and Application*. Amsterdam, The Netherlands: North-Holland, 1979, pp. 3–18.
[31] K. Koffka, *Principles of Gestalt Psychology*. San Diego, CA, USA: Harcourt, 1967.
[32] M. S. Ishwarya and C. A. Kumar, “Quantum aspects of high dimensional conceptual space: A model for achieving consciousness,” *Cogn. Comput.*, vol. 12, pp. 563–576, Jan. 2020.

Weihua Xu received the M.Sc. degree in mathematics from the School of Mathematics and Information Sciences, Guangxi University, Nanning, China, in 2004, and the Ph.D. degree in mathematics from the School of Sciences, Xi’an Jiaotong University, Xi’an, China, in 2007.

He is currently a Professor with the College of Artificial Intelligence, Southwest University, Chongqing, China. He serves on the editorial boards of several international journals, and has authored and coauthored four monographs and more than 180 articles in international journals. His current research interests include granular computing, approximate reasoning, fuzzy sets, rough sets, concept lattices, cognitive computing, and evolutionary computing.

Doudou Guo received the M.Sc. degree in computer science from the School of Computer Science and Information Engineering, Harbin Normal University, Harbin, China, in 2021. He is currently working toward the Ph.D. degree with the College of Artificial Intelligence, Southwest University, Chongqing, China.

His current research interests include concept learning, cognitive computing, granular computing, and fuzzy sets.

Yuhua Qian received the M.S. and Ph.D. degrees in computers with applications from Shanxi University, Taiyuan, China, in 2005 and 2011, respectively.

He is currently a Director with the Institute of Big Data Science and Industry, Shanxi University, where he is also a Professor with the Key Laboratory of Computational Intelligence and Chinese Information Processing, Ministry of Education. He has authored more than 100 articles in international journals. His research interests include artificial intelligence, machine learning, and machine vision.

Weiping Ding received the Ph.D. degree in computer science from the Nanjing University of Aeronautics and Astronautics, Nanjing, China, in 2013.

He is currently a Full Professor with the School of Information Science and Technology, Nantong University, Nantong, China. He was a Visiting Scholar with the University of Lethbridge, Alberta, Canada, in 2011. From 2014 to 2015, he was a Postdoctoral Researcher with the Brain Research Center, National Chiao Tung University, Hsinchu, Taiwan. In 2016, he was a Visiting Scholar with the National University of Singapore, Singapore. From 2017 to 2018, he was a Visiting Professor with the University of Technology Sydney, Ultimo, NSW, Australia. He has authored and coauthored more than 130 scientific articles in refereed international journals. He has held 23 approved invention patents. He has co-authored two books. His ten authored/coauthored papers have been selected as ESI Highly Cited Papers. His main research interests include deep neural networks, multimodal machine learning, granular data mining, and medical images analysis.

Dr. Ding served/serves on the Editorial Board for *Knowledge-Based Systems*, *Information Fusion*, *Engineering Applications of Artificial Intelligence*, and *Applied Soft Computing*. He served/serves as an Associate Editor for IEEE Transactions on Neural Networks and Learning Systems, IEEE Transactions on Fuzzy Systems, IEEE/CAA Journal of Automatica Sinica, IEEE Transactions on Intelligent Transportation Systems, *Information Sciences*, *Neurocomputing*, *Swarm and Evolutionary Computation*, and so on.