Two-Way Concept-Cognitive Learning via Concept Movement Viewpoint

Weihua Xu, Doudou Guo, Jusheng Mi, Yuhua Qian, *Member, IEEE*, Keyin Zheng, and Weiping Ding

***Abstract*— Representation and learning of concepts are critical problems in data science and cognitive science. However, the existing research about concept learning has one prevalent disadvantage: incomplete and complex cognitive. Meanwhile, as a practical mathematical tool for concept representation and concept learning, two-way learning (2WL) also has some issues leading to the stagnation of its related research: the concept can only learn from specific information granules and lacks a concept evolution mechanism. To overcome these challenges, we propose the two-way concept-cognitive learning (TCCL) method for enhancing the flexibility and evolution ability of 2WL for concept learning. We first analyze the fundamental relationship between two-way granule concepts in the cognitive system to build a novel cognitive mechanism. Furthermore, the movement three-way decision (M-3WD) method is introduced to 2WL to study the concept evolution mechanism via the concept movement viewpoint. Unlike the existing 2WL method, the primary consideration of TCCL is two-way concept evolution rather than information granules transformation. Finally, to interpret and help understand TCCL, an example analysis and some experiments on various datasets are carried out to demonstrate our method's effectiveness. The results show that TCCL is more flexible and less time-consuming than 2WL, and meanwhile, TCCL can also learn the same concept as the latter method in concept learning. In addition, from the perspective of concept learning ability, TCCL is more generalization of concepts than the granule concept cognitive learning model (CCLM).**

***Index Terms*— Concept-cognitive learning (CCL), concept evolution, granular computing, three-way decision, two-way learning (2WL).**

I. Introduction

THE development of big data has opened up a whole new era for artificial intelligence [7], [20], [28]. As an essential basis to support artificial intelligence, the theory and method of big-data analysis are related to the formulation of man–machine intelligence. The data–information–knowledge–wisdom (DIKW) hierarchy is one of the fundamental concepts in the big-data analysis theory. That is, data creates information, information creates knowledge, and knowledge creates wisdom [29]. Specifically, the DIKW hierarchy emphasizes discovering the correct information that human-being from data can use, and finally, using the obtained information to guide human decision-making. As is well known, the knowledge acquisition process is inseparable from the representation and learning of concepts [14], [18], [32]. Currently, cognitive learning for concepts is emerging in artificial intelligence and cognitive science [10], [15], [25].

Concept learning and concept cognitive are two emerging issues in machine learning and cognitive science. Note that people learning a new concept can often generalize successfully from very few examples, yet machine-learning approaches typically require tens or hundreds of examples to perform with similar accuracy [3]. Therefore, concept-cognitive learning (CCL) theory emerges that simulates the human cognitive process by integrating concept learning and concept cognitive. Concepts can be learned from objects/attributes and acquired through a pair of cognitive operators to describe the relationship between objects and attributes, namely concept generation. In addition, the acquired concepts can also be cognitive through a specific learning model, that is, concept evolution. However, some existing CCL system lacks concept generation and evolution capability. For instance, Xu et al. [26] and Xu and Li [27] propose the two-way learning (2WL) system to learn concepts, while it lacks the concept evolution ability due to the main forces on the granule description and transformation mechanism. Moreover, Shi et al. [31] study the concept-cognitive learning method via concept space learning, which also lacks the concept generation ability due to its primary focus on constructing concept space. Hence, a novel CCL system is required to promote the concept generation and evolution capability of cognitive learning, which is one of the main goals of this article.

Moreover, we noticed that 2WL methods [26], [27] still have some problems: 1) the sufficient and necessary granule concept could only be obtained from the necessary or sufficient granule concept; 2) the two-way granule concept cannot be obtained from the sufficient and necessary granules; and 3) the conclusion that the number of sufficient and necessary granules is less than 16 is inaccurate. Although 2WL is a classic and effective concept-cognitive learning model, these problems lead to the stagnation of its related research. Therefore, pointing out and solving these problems is another motivation for this article.

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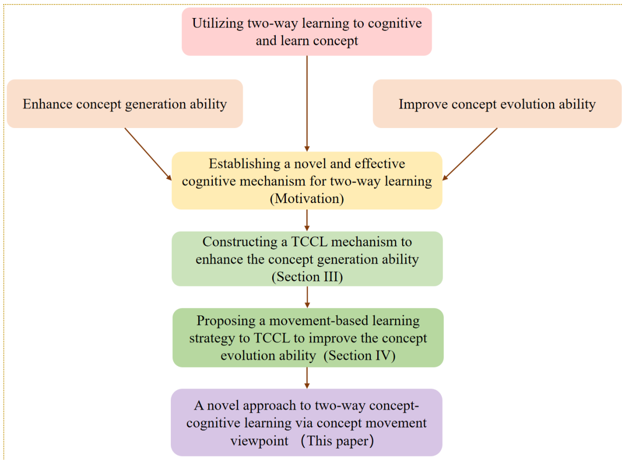

Fig. 1. Block diagram of the proposed approach.

The concept-cognitive learning theory establishes rigorous cognitive models and provides new semantics interpretation for knowledge discovery in the learning process. In other words, knowledge of the natural world can be embodied through these particular conceptual structures, and wisdom can be learned through these particular conceptual structures. All in all, CCL has been investigated from various aspects. Nevertheless, flexibility and evolution ability in concept learning still needs to be explored. Note that the core idea of movement-based three-way decision [4], [5], [6] is to move objects among three regions by specific action or strategy and then complete evolutionary learning of different regions to form a new tri-partition. This article applies the naive idea of M-3WD to the two-way concept-cognitive learning (TCCL) model to think about its cognitive mechanism. In this mechanism, we use the existing methods and expressions of M-3WD to learn more concepts to achieve concept evolution, which is concept movement. Therefore, the last motivation in the current article is how to integrate this idea into CCL systems.

In this article, to cope with these limitations, a novel CCL system (TCCL) is proposed via a concept movement view. The block diagram of the proposed approach is shown in Fig. 1. The main contributions of this article are as follows.

1) We propose a TCCL method to address the incomplete and complex cognitive problem of 2WL. The core idea is to introduce the concept-cognitive mechanism and M-3WD model into 2WL, simultaneously enhancing the ability of concept generation and concept evolution.
2) We formulate a novel cognitive mechanism for 2WL by exploring the relationship between different granule concepts. This mechanism can be more flexible and less time-consuming to learn concepts from the given clue, and meanwhile, we verify that our method achieves better concept learning performance than other methods.
3) We present a concept evolution strategy from the movement perspective to evolve concept space. One can acquire more concepts to form knowledge and provide a new research view for knowledge-discover and decision-making. Experimental results on 12 datasets show the effectiveness of the proposed strategy.

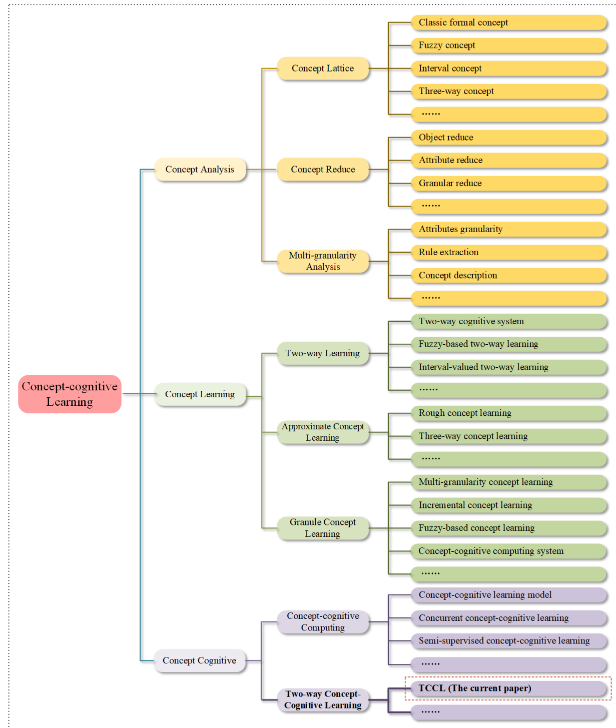

Fig. 2. Categorization of CCL.

This article is organized as follows. Section II briefly reviews the development of concept-cognitive learning and some challenges of 2WL. Section III presents a novel cognitive mechanism based on 2WL (i.e., TCCL). A movement-based learning strategy for TCCL is presented in Section IV. Section V gives an example analysis. The experimental analysis is given in Section VI. Finally, this article is concluded with further work in Section VII.

II. Related Work

In this section, we use Fig. 2 to categorize various concept models under the concept-cognitive learning theory, where the contributions of this article are highlighted in bold font. The diagram may be viewed as a list of examples rather than an exhaustive summary.

During the past few years, we have witnessed a growing interest in concept-cognitive learning. The theory is motivated by a particular cognitive mechanism to explore the learning model of concepts from data, which is a valuable data analysis and knowledge discovery method. Currently, CCL theory focuses on three aspects: 1) concept analysis method; 2) concept learning strategy; and 3) concept cognition mechanism. Recent studies along these lines of thought have fostered many concept-cognitive learning models. Regarding concept analysis, scholars mainly carried out a series of studies in concept lattice, concept reduction, and multigranularity analysis. Zhang and Xu [24] first investigate concepts from the unknown through a pair of cognitive operators. Yan et al. [8], [9] combines the three-way decision with the partial-order structure to study the learning and cognitive of concept learning, and a three-way concept (TWC) based on apposition and subposition is constructed in [22].

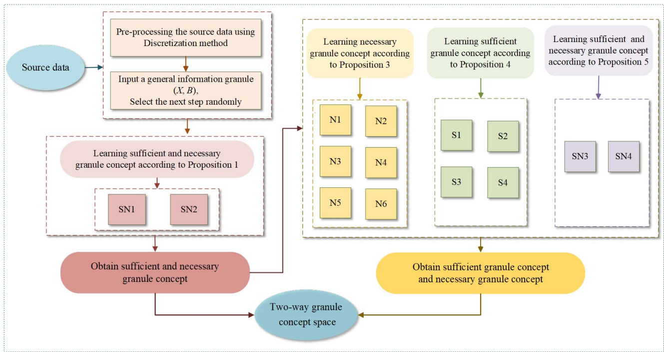

Fig. 3. Processes of the novel cognitive mechanism based on 2WL.

In terms of concept learning, some research about 2WL, approximate concept learning, and granule concept learning has also attracted wide attention. For instance, Li et al. [12], [13] discuss concept learning via granular computing from the cognitive view and study cognitive processes whose aims are to deal with the problem of learning approximate cognitive concepts. Zhang et al. [21] propose a solution to this issue by introducing an incremental concept tree representation. Horzyk et al. [11], Li et al. [1], Yuan et al. [19], and Hatwagner et al. [16] focus on continual learning and reduction of concepts.

Especially, the research on concept-cognitive computing in machine learning are emerging in recent years, Mi et al. [30] focus on concept generalization and map all samples into different concept spaces containing various concepts formed by a Galois connection; Zhao et al. [33] mainly discuss cognitive concept learning from incomplete information and simulating the cognitive processes via three types of similarities to learn the granule concept.

A 2WL system is a cognitive process that learns from useless information [26], [27]. Meanwhile, we note that the 2WL system has some problems: 1) the necessary and sufficient granule concept has to obtain from the necessary or sufficient granule concept; and 2) the two-way granule concept cannot obtain from the necessary and sufficient granule concept.

Based on the above-mentioned issues, the relevant study process of 2WL stagnates, and only a few research findings are available. Note that the research of concept-cognitive learning is at an early stage. Although there have been many significant achievements, it also appears very important to improve and enrich the area of CCL from various theories, frameworks, models, and viewpoints. Following the path of 2WL, we investigate some noteworthy issues in the study of 2WL and describe a basic idea of TCCL for these issues.

III. Novel Cognitive Mechanism Based on 2WL

As is well known, 2WL can transform a given information granule into two-way granule concepts. Meanwhile, we find that the sufficient and necessary granule concept can also be transformed into a sufficient and necessary granule concept. Hence, this section investigates the granule concept learning method by analyzing the sufficient, necessary, and sufficient and necessary granule concept relations.

The novel cognitive mechanism (i.e., TCCL mechanism) depicting the cognitive process of three granule concepts is shown in Fig. 3. It consists of three-stage: 1) the first stage is preprocessing source data using the discretization method and inputting an information granule; 2) we can learn sufficient and necessary granule concepts according to Proposition 1; and 3) we can learn the new granule concepts through the granule concept stored in the second stage via Propositions 3–5. Thus, the granule concept space can be learned according to the above method in Fig. 3.

A. Two-Way Learning

A formal context is a triple $F = (U, A, I)$, where U and A are two nonempty finite sets of object and attribute, respectively, and I is a binary relation on $U \times A$. In addition, a pair of set-valued mappings $L : 2^U \to 2^A$ and $H : 2^A \to 2^U$ are called concept cognitive operates if it satisfies the properties in [26], and they are abbreviated as L and H, respectively.

Definition 1: Let $L_1 = P(U)$ and $L_2 = P(A)$ be two complete lattices, L and H be two cognitive operators (i.e., (L_1, L_2, L, H) be a cognitive system). For any $X \in L_1$ and $B \in L_2$, denote

$$\mathcal{G}_1 = \{(X, B) | B \leqslant L(X), X \leqslant H(B)\} \quad (1)$$

$$\mathcal{G}_2 = \{(X, B) | L(X) \leqslant B, H(B) \leqslant X\}. \quad (2)$$

1) If $(X, B) \in \mathcal{G}_1$, then (X, B) is a necessary granule concept of (L_1, L_2, L, H). Simultaneously, $\mathcal{G}_1$ is a necessary granule concept space.
2) If $(X, B) \in \mathcal{G}_2$, then (X, B) is a sufficient granule concept of (L_1, L_2, L, H). Simultaneously, $\mathcal{G}_2$ is a sufficient granule concept space.
3) If $(X, B) \in \mathcal{G}_1 \cap \mathcal{G}_2$, that is, (X, B) satisfies $B = L(X)$ and $X = H(B)$, then (X, B) is a sufficient and necessary granule concept of (L_1, L_2, L, H). Simultaneously, $\mathcal{G}_1 \cap \mathcal{G}_2$ is a sufficient and necessary granule concept space.

$\leqslant$ is a quasi-order relationship.

From Definition 1, we only consider the situation that there exist three granule concept spaces in (L_1, L_2, L, H). However, $(X, B) \notin \mathcal{G}_1 \cup \mathcal{G}_2$ is not a granule concept of (L_1, L_2, L, H). Moreover, if granule concepts do not exist at the beginning of (L_1, L_2, L, H). The approaches to learning these granule concepts are as follows.

Property 1: (see [26]) Let (L_1, L_2, L, H) be a cognitive system, $\mathcal{G}_1$ be a necessary granule concept space, and $\mathcal{G}_2$ be a sufficient granule concept space. If $X \in L_1$ and $B \in L_2$, then

1) $(X \wedge H(B), B \vee L(X)) \in \mathcal{G}_1$
2) $(X \vee H(B), B \wedge L(X)) \in \mathcal{G}_1$
3) $(H(B), B \wedge L(X)) \in \mathcal{G}_1$
4) $(X \wedge H(B), L(X)) \in \mathcal{G}_1$
5) $(HL(X), B \wedge L(X)) \in \mathcal{G}_1$
6) $(X \wedge H(B), LH(B)) \in \mathcal{G}_1$
7) $(X \vee H(B), LH(B)) \in \mathcal{G}_2$
8) $(HL(X), B \vee L(X)) \in \mathcal{G}_2$.

Property 2: (see [26]) Let (L_1, L_2, L, H) be a cognitive system. If $(X_1, B_1) \in \mathcal{G}_1$ and $(X_2, B_2) \in \mathcal{G}_2$, then

1) $(X_1 \vee H(B_1), L(X_1 \vee H(B_1))) \in \mathcal{G}_1 \cap \mathcal{G}_2$
2) $(H(B_1 \vee L(X_1)), B_1 \vee L(X_1)) \in \mathcal{G}_1 \cap \mathcal{G}_2$
3) $(X_2 \wedge H(B_2), L(X_2 \wedge H(B_2))) \in \mathcal{G}_1 \cap \mathcal{G}_2$
4) $(H(B_2 \wedge L(X_2)), B_2 \wedge L(X_2)) \in \mathcal{G}_1 \cap \mathcal{G}_2$.

Thus, for (L_1, L_2, L, H) be a cognitive system, we define "$\vee$" and "$\wedge$" are operators of cognitive system, and

$$(X_1, B_1) \wedge (X_2, B_2) = (X_1 \wedge X_2, LH(B_1 \vee B_2)) \quad (3)$$
$$(X_1, B_1) \vee (X_2, B_2) = (HL(X_1 \vee X_2), B_1 \wedge B_2). \quad (4)$$

B. Sufficient and Necessary Granule Concept

Note that there are two issues in 2WL [26], [27]: 1) the sufficient and necessary granule concept can only be obtained by necessary or sufficient granule concepts way (i.e., Property 2); and 2) for arbitrary information granules, through 2WL method, the number of sufficient and necessary granule concept no more than 16 in theory.

In this section, we first show some new standpoints, including 1) sufficient and necessary granule concepts can be learned directly from arbitrary information granules and 2) the number of sufficient and necessary granule concepts is no more than six (strictly speaking, no more than two in some cases). Then, we can further present the new notion and proposition.

Let (L_1, L_2, L, H) be a cognitive system, the granule concept (X, B) is a sufficient and necessary granule concept, if any $X \in L_1$ and $B \in L_2$, $(X, B) \in \mathcal{G}_1 \cap \mathcal{G}_2$.

Proposition 1: Let (L_1, L_2, L, H) be a cognitive system, $\mathcal{G}_1$ be a necessary granule concept space, and $\mathcal{G}_2$ be a sufficient granule concept space, $\mathcal{G}_1 \cap \mathcal{G}_2$ be a sufficient and necessary granule concept space. If $X \in L_1$ and $B \in L_2$, then

1) $(HL(X), L(X)) \in \mathcal{G}_1 \cap \mathcal{G}_2$
2) $(H(B), LH(B)) \in \mathcal{G}_1 \cap \mathcal{G}_2$.

Proof: It is directly obtained from the basic notions in formal concept analysis and Definition 1.

Note that Proposition 1 directly states that $(HL(X), L(X))$ and $(H(B), LH(B))$ are two sufficient and necessary granule concepts of the granule concept space $\mathcal{G}_1 \cap \mathcal{G}_2$. Next, we will examine the specific relationship between them.

Proposition 2: Let (L_1, L_2, L, H) be a cognitive system. For arbitrary information granule $(X, B) \in \mathcal{G}_1 \cup \mathcal{G}_2$, there is only one sufficient and necessary granule concept which is itself [i.e., (X, B)]; Otherwise, we have two sufficient and necessary granule concepts, that is, $(HL(X), L(X))$ and $(H(B), LH(B))$.

Proof. To prove this proposition, we divide it into two steps as follows.

1) For one sufficient and necessary granule concept. Because (L_1, L_2, L, H) is a cognitive system, it is immediate from Properties 1 and 2.
2) For two sufficient and necessary granule concepts. Because (L_1, L_2, L, H) is a cognitive system, from Definition 1 and Property 1, we have three cases of granule concept: 1) $\mathcal{G}_1$; 2) $\mathcal{G}_2$; and 3) $(\mathcal{G}_1 \cup \mathcal{G}_2)^c$, where $(\cdot)^c$ is the complement. Then, we divide it into three cases to prove it.
   a) If $(X, B) \in \mathcal{G}_1$, from Definition 1, we have $X \leqslant H(B)$ and $B \leqslant L(X)$. Thus, from Property 2-1), we have $X \vee H(B) = H(B)$, $L((X \vee H(B)) = LH(B)$; from Property 2-2), we have $B \vee L(X) = L(X)$, $H(B \vee L(X)) = HL(X)$. Hence, two sufficient and necessary granule concepts are $[HL(X), L(X)]$ and $[H(B), LH(B)]$.
   b) If $(X, B) \in \mathcal{G}_2$, similarly, we can prove this case.
   c) If $(X, B) \in (\mathcal{G}_1 \cup \mathcal{G}_2)^c$, it is immediate from Definition 1 and Proposition 1.

By (i) and (ii), this proposition can be proved.

Intuitively, Propositions 1 and 2 show that $[HL(X), L(X)]$ and $[H(B), LH(B)]$ are two sufficient and necessary granule concepts in the 2WL system. For any $X \in L_1$ and $B \in L_2$, if (X, B) is sufficient and necessary granule concept, then $(X, B) = (HL(X), L(X)) = (H(B), LH(B))$. According to the above discussion, we have the corollary as follows.

Corollary 1: Let (L_1, L_2, L, H) be a cognitive system. For arbitrary information granule $(X, B) \in \mathcal{G}_1 \cup \mathcal{G}_2$, the number of the sufficient and necessary granule concepts is no more than two.

According to the above discussion, one can directly learn sufficient and necessary granule concepts from arbitrary information granules through our method. The details are shown in Algorithm 1.

Algorithm 1 Learn Granule Concept From Arbitrary Information Granule

Input: Arbitrary information granule (X, B), a formal context $F = (U, A, I)$, granule concept space $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$.
Output: Necessary granule concept, sufficient granule concept, sufficient and necessary granule concept: (X^1, B^1), (X^2, B^2), (X^3, B^3).

1: Let $\mathcal{G}_1 = \emptyset$, $\mathcal{G}_2 = \emptyset$, $\mathcal{G}_3 = \emptyset$;
2: **while** $(X, B) \subseteq \mathcal{G}_1 \cup \mathcal{G}_2$ **do**
3: Learn necessary granule concept (X^1, B^1) from (X, B), $({X^1}_1, {B^1}_1), ({X^1}_2, {B^1}_2), \ldots, ({X^1}_m, {B^1}_m)$, $m \leqslant 6$, by four methods according to Property 1;
4: Learn sufficient granule concept (X^2, B^2) from (X, B), $({X^2}_1, {B^2}_1), \cdots, ({X^2}_m, {B^2}_m)$, $m \leqslant 2$, by four methods according to Property 1;
5: Learn sufficient and necessary granule concept (X^3, B^3) from (X, B), $({X^3}_1, {B^3}_1), \cdots, ({X^3}_m, {B^3}_m)$, $m \leqslant 2$, by two methods according to Proposition 1;
6: $\mathcal{G}_1 \leftarrow (X^1, B^1)$; $\mathcal{G}_2 \leftarrow (X^2, B^2)$; $\mathcal{G}_3 \leftarrow (X^3, B^3)$;
7: **end while**

Given a formal context $F = (U, A, I)$. The cardinality of objects and attributes is denoted by $|U|$ and $|A|$, respectively. The cardinality of objects and attributes of arbitrary information granule (X, B) is denoted by $|X|$ and $|B|$, respectively. Next, we can analyze the time complexity of Algorithm 1. Running step 1, take $O(1)$ due to initialized setting. In steps 2–7, its running time is decided by the while statement. Thus, the running time complexity of Algorithm 1 takes $O(|U| \cdot (|X| + |AT|))$.

C. Granule Concept in Cognitive Systems

Intrinsically, a 2WL mechanism is the cognitive systems begin to acquire concepts from the unknown. Note that 2WL can effectively transform arbitrary information granules into sufficient and necessary granule concepts, necessary granule concepts, and sufficient granule concepts. However, it is not suitable for transforming sufficient and necessary granule concepts into other granule concepts (see Proposition 2). In other words, when the information granule is sufficient and necessary granule concept, Property 1 does not hold.

Inspired by the 2WL mechanism, we put forward a novel cognitive mechanism for transforming sufficient and necessary granule concepts into necessary or sufficient granule concepts.

Case 1: The method to transform a pair of sufficient and necessary granule concepts into necessary granule concepts can be represented in the following.

Proposition 3: Let (L_1, L_2, L, H) be a cognitive system, $\mathcal{G}_1 \cap \mathcal{G}_2 = \{(HL(X), L(X)) | X \in L_1\} \cup \{(H(B), LH(B)) | B \in L_2\}$ be a sufficient and necessary granule concept space. Then the following statements hold.

1) $(HL(X), L(X) \wedge LH(B)) \in \mathcal{G}_1$.
2) $(HL(X) \wedge H(B), L(X)) \in \mathcal{G}_1$.
3) $(H(B), LH(B) \wedge L(X)) \in \mathcal{G}_1$.
4) $(H(B) \wedge HL(X), LH(B)) \in \mathcal{G}_1$.
5) $(HL(X) \vee H(B), L(X) \wedge LH(B)) \in \mathcal{G}_1$.
6) $(H(B) \wedge HL(X), LH(B) \vee L(X)) \in \mathcal{G}_1$.

Proof: 1) Because (L_1, L_2, L, H) is a cognitive system, from the basic notions in Section III-A and Definition 1, we have $L(HL(X)) = L(X) \geqslant L(X) \wedge LH(B)$ and $H(L(X) \wedge LH(B)) \geqslant HL(X) \vee H(B) \geqslant HL(X)$.

Thus, $(HL(X), L(X) \wedge LH(B)) \in \mathcal{G}_1$.

The way to prove items 2)–6) is similar to 1).

This Proposition is proven.

Case 2: The method to transform a pair of sufficient and necessary granule concepts into sufficient granule concepts can be represented in the following.

Proposition 4: Let (L_1, L_2, L, H) be a cognitive system, $\mathcal{G}_2$ be a sufficient granule concept space, $\mathcal{G}_1 \cap \mathcal{G}_2 = \{(HL(X), L(X)) | X \in L_1\} \cup \{(H(B), LH(B)) | B \in L_2\}$ be a sufficient and necessary granule concept space. Then the following statements hold.

1) $(HL(X), L(X) \vee LH(B)) \in \mathcal{G}_2$.
2) $(HL(X) \vee H(B), L(X)) \in \mathcal{G}_2$.
3) $(H(B), LH(B) \vee L(X)) \in \mathcal{G}_2$.
4) $(H(B) \vee HL(X), LH(B)) \in \mathcal{G}_2$.

Proof: 1) Because (L_1, L_2, L, H) is a cognitive system, from the basic notions in Section III-A and Definition 1, we have $H(L(X) \vee LH(B)) = HL(X) \wedge H(B) \leqslant HL(X)$ and $L(HL(X)) = L(X) \leqslant L(X) \vee LH(B))$.

Thus, $(HL(X), L(X) \vee LH(B)) \in \mathcal{G}_2$.

The way to prove items 2)–4) is similar to 1).

This proposition is proven.

Case 3: The method to learn a pair of sufficient and necessary granule concepts from itself can be represented in the following.

Proposition 5: Let (L_1, L_2, L, H) be a cognitive system, $\mathcal{G}_1 \cap \mathcal{G}_2 = \{(HL(X), L(X)) | X \in L_1\} \cup \{(H(B), LH(B)) | B \in L_2\}$ be a sufficient and necessary granule concept space. Then the following statements hold.

1) $(H(L(X) \wedge LH(B)), L(X) \wedge LH(B)) \in \mathcal{G}_1 \cap \mathcal{G}_2$.
2) $(HL(X) \wedge H(B), L(HL(X) \wedge H(B))) \in \mathcal{G}_1 \cap \mathcal{G}_2$.

Proof. 1) Because (L_1, L_2, L, H) is a cognitive system, from the basic notions in Section III-A and Definition 1, we have $L(H(L(X) \wedge LH(B))) = LH(L(X \vee H(B)) = LHL(X \vee H(B)) = L(X \vee H(B)) = L(X) \wedge LH(B)$ and $H(L(X) \wedge LH(B)) = H(L(X) \wedge LH(B))$.

Thus, $(H(L(X) \wedge LH(B)), L(X) \wedge LH(B)) \in \mathcal{G}_1 \cap \mathcal{G}_2$.

2) This item can be obtained similarly.

This proposition is proven.

Based on the above discussion, a novel cognitive mechanism based on 2WL of granule-concept (including arbitrary information granule can transform to sufficient and necessary granule concept, and sufficient and necessary granule concept also can transform to necessary or sufficient granule concept) is presented. The details of the novel cognitive mechanism are shown in Algorithm 2.

Algorithm 2 Learn Granule Concept From Sufficient and Necessary Granule Concept

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Input: Arbitrary information granule (X, B).

Output: Necessary granule concept, sufficient granule concept, sufficient and necessary granule concept: (X^1, B^1), (X^2, B^2), (X^3, B^3); granule concept space: $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$.

1: Construct a sufficient and necessary granule concept space $\mathcal{G}_3$ by Algorithm 1;
2: Let $\mathcal{G}_1 = \emptyset$, $\mathcal{G}_2 = \emptyset$;
3: **for all** $(X, B) \subseteq \mathcal{G}_3$ **do**
4: Learn necessary granule concept (X^1, B^1) from (X, B), $({X^1}_1, {B^1}_1), ({X^1}_2, {B^1}_2), \ldots, ({X^1}_m, {B^1}_m)$, $m \leqslant 6$ by four methods according to Proposition 3;
5: Learn sufficient granule concept (X^2, B^2) from (X, B), $({X^2}_1, {B^2}_1), ({X^2}_2, {B^2}_2), \ldots, ({X^2}_m, {B^2}_m)$, $m \leqslant 4$ by four methods according to Proposition 4;
6: Learn sufficient and necessary granule concept (X^3, B^3) from (X, B), $({X^3}_1, {B^3}_1), \cdots, ({X^3}_m, {B^3}_m)$, $m \leqslant 2$ according to Proposition 5;
7: **end for**
8: $\mathcal{G}_1 \leftarrow (X^1, B^1)$; $\mathcal{G}_2 \leftarrow (X^2, B^2)$. $\mathcal{G}_3 \leftarrow (X^3, B^3)$.

Now, we can analyze the time complexity of Algorithm 2. The time complexity of step 1 is $O(|U| \cdot (|X| + |AT|))$. Running step 2, take $O(1)$ due to initialized setting. In steps 3–7, its running time is decided by the for-loop, that is, $O(|U| \cdot (|X| + |AT|))$. Running step 8, take $O(1)$. Thus, the running time complexity of Algorithm 2 takes $O(|U| \cdot (|X| + |AT|))$.

IV. Movement-Based Learning Strategy for TCCL

In this section, we mainly integrate a movement-based learning strategy into TCCL (i.e., TCCL with the movement-based learning strategy, M-TCCL) to further explore the concept evolution mechanism.

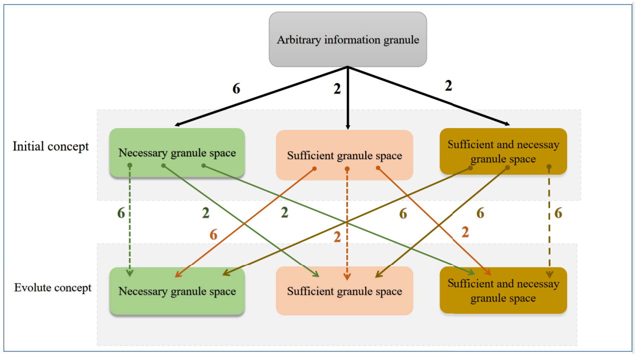

Fig. 4. Evolution framework of TCCL.

A. Movement Viewpoint

In Section III, we have demonstrated that the TCCL method can learn the two-way granule concept from arbitrary information granules. It is not difficult to find that the cognitive mechanism of TCCL is consistent with the basic idea of M-3WD. Thus, this section integrates the M-3WD method with TCCL and studies a novel two-way concept-cognitive mechanism via a concept movement view. Furthermore, from the perspective of movement [5], [6], we can get an evolution framework of TCCL as shown in Fig. 4.

From the perspective of movement, we focus on the evolution of the granule concept from one granule concept space to another, that is, the position of a new granule concept in TCCL. The position function formally defines as follows.

Given a cognitive system (L_1, L_2, L, H), a weak tri-partition $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1 \cap \mathcal{G}_2\}$ of a whole $\mathcal{G}_1 \cup \mathcal{G}_2$, let a position function $\mathcal{G}^p : \mathcal{G} \to \mathcal{G}^p$ produce the position of an granule concept $(X, B) \in \mathcal{G}$, where $\mathcal{G}^p = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$. The function is defined as follows:

$$\mathcal{G}^p((X,B)) = \begin{cases} \mathcal{G}^1, & (X,B) \in \mathcal{G}_1 \\ \mathcal{G}^2, & (X,B) \in \mathcal{G}_2 \\ \mathcal{G}^3, & (X,B) \in \mathcal{G}_1 \cap \mathcal{G}_2. \end{cases} \quad (5)$$

The movement of a granule concept can be defined as an evolution of its position caused by a pair of logical operators (i.e., $\vee$ and $\wedge$). The overall movement leads to a movement from one tri-partition $\mathcal{G}$ to another $\mathcal{G}'$, which is the movement viewpoint. Thus, all results of cognitive learning lead to a new granule concept space $\mathcal{G}' = \{\mathcal{G}'_1, \mathcal{G}'_2, \mathcal{G}'_1 \cap \mathcal{G}'_2\}$.

Given a weak tri-partition $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1 \cap \mathcal{G}_2\}$, L and H is a pair of cognitive operators. The movement of tri-partition caused by cognitive operators L and H is denoted as $\mathcal{G} \rightsquigarrow_{LH} \mathcal{G}'$. The corresponding movement of the position of $(X, B) \in \mathcal{G}$ is denoted as $\mathcal{G}^p((X,B)) \rightsquigarrow_{LH} \mathcal{G}^p((X,B))$.

When applying the logical operator (i.e., $\vee$ and $\wedge$) and cognitive operator (i.e., L and H), any concept $(X, B) \in \mathcal{G}$ has three possible movements that are, staying in the same granule concept space or transferred to either of the other two. Thus, for the movements of concept, there are nine possibilities as follows:

$$\begin{aligned} &\mathcal{G}^1 \rightsquigarrow_{LH} \mathcal{G}^1, \mathcal{G}^1 \rightsquigarrow_{LH} \mathcal{G}^2, \mathcal{G}^1 \rightsquigarrow_{LH} \mathcal{G}^3 \\ &\mathcal{G}^2 \rightsquigarrow_{LH} \mathcal{G}^1, \mathcal{G}^2 \rightsquigarrow_{LH} \mathcal{G}^2, \mathcal{G}^2 \rightsquigarrow_{LH} \mathcal{G}^3 \\ &\mathcal{G}^3 \rightsquigarrow_{LH} \mathcal{G}^1, \mathcal{G}^3 \rightsquigarrow_{LH} \mathcal{G}^2, \mathcal{G}^3 \rightsquigarrow_{LH} \mathcal{G}^3 \end{aligned} \quad (6)$$

where for $\mathcal{G}^i \rightsquigarrow_{LH} \mathcal{G}^j (i, j \in \{1,2,3\})$, $\mathcal{G}^i$ represents the position of (X, B) before movement, and $\mathcal{G}^j$ represents the position after movement, which is the process of concept movement.

Inspired by Fig. 4 and the 2WL system (L_1, L_2, L, H), we proposed a novel learning strategy for TCCL from a movement view. Cases 1–3 in Section III-C mainly discuss the method to learn granule concepts from sufficient and necessary granule concepts based on the two cognitive operators. In this section, we mainly talk about the initial granule concept in TCCL. Similar to the initial information X and B, X^L and B^H represent extent and intent, respectively. Thus, we have one of the following three cases.

Case 1': Learning the necessary granule concept from the sufficient and necessary granule concept can be represented in the following.

Proposition 6: Let (L_1, L_2, L, H) be a cognitive system, $\mathcal{G}_1 \cap \mathcal{G}_2 = \{(HL(X^L), L(X^L))|X \in L_1\} \cup \{(H(B^H), LH(B^H))|B \in L_2\}$ be a sufficient and necessary granule concept space. Then the following statements hold.

1) $(HL(X), L(X \vee H(B))) \in \mathcal{G}_1$.
2) $(HL(X \wedge H(B)), L(X)) \in \mathcal{G}_1$.
3) $(HL(X \wedge H(B)), L(X \vee H(B)) \in \mathcal{G}_1$.
4) $(H(B \vee L(X)), LH(B)) \in \mathcal{G}_1$.
5) $(H(B \vee L(X)), LH(B \wedge L(X))) \in \mathcal{G}_1$.
6) $(H(B), LH(B \wedge L(X))) \in \mathcal{G}_1$.

Proof. The proof can be derived by the basic notions in Section III-A, Property 1, and Definition 1.

Case 2': Learning sufficient granule concepts from a pair of sufficient and necessary granule concepts can be represented in the following.

Proposition 7: Let (L_1, L_2, L, H) be a cognitive system, $\mathcal{G}_2$ be a sufficient granule concept space, $\mathcal{G}_1 \cap \mathcal{G}_2 = \{(HL(X^L), L(X^L))|X \in L_1\} \cup \{(H(B^H), LH(B^H))|B \in L_2\}$ be a sufficient and necessary granule concept space. Then the following statements hold.

1) $(HL(X \vee H(B)), L(X)) \in \mathcal{G}_2$.
2) $(HL(X \vee H(B)), L(X \wedge H(B))) \in \mathcal{G}_2$.
3) $(HL(X), L(X \wedge H(B))) \in \mathcal{G}_2$.
4) $(H(B \wedge L(X)), LH(B)) \in \mathcal{G}_2$.
5) $(H(B \wedge L(X)), LH(B \vee L(X))) \in \mathcal{G}_2$.
6) $(H(B), LH(B \vee L(X))) \in \mathcal{G}_2$.

Proof: The proof can be derived by the basic notions in Section III-A, Property 1, and Definition 1.

Case 3': Learning sufficient and necessary granule concepts from a pair of sufficient and necessary granule concepts can be represented in the following.

Proposition 8: Let (L_1, L_2, L, H) be a cognitive system, $\mathcal{G}_1 \cap \mathcal{G}_2 = \{(HL(X^L), L(X^L))|X \in L_1\} \cup \{(H(B^H), LH(B^H))|B \in L_2\}$ be a sufficient and necessary granule concept space. Then the following statements hold.

1) $(HL(X), L(X)) \in \mathcal{G}_1 \cap \mathcal{G}_2$.
2) $(H(B), LH(B)) \in \mathcal{G}_1 \cap \mathcal{G}_2$.
3) $(HL(X \wedge H(B)), L(X \wedge H(B))) \in \mathcal{G}_1 \cap \mathcal{G}_2$.
4) $(HL(X \vee H(B)), L(X \vee H(B))) \in \mathcal{G}_1 \cap \mathcal{G}_2$.
5) $(H(B \vee L(X)), LH(B \vee L(X))) \in \mathcal{G}_1 \cap \mathcal{G}_2$.
6) $(H(B \wedge L(X)), LH(B \wedge L(X))) \in \mathcal{G}_1 \cap \mathcal{G}_2$.

Proof. The proof can be derived by the basic notions in Section III-A, Definition 1, and Proposition 1.

Based on Propositions 6–8, we need to point out that X^L and B^H represent no more than one cognitive operation L or H, respectively. It is easy to prove that $X^L \in \{X, X \vee H(B), X \wedge H(B)\}$ and $B^H \in \{B, B \vee L(X), B \wedge L(X)\}$, where $X \in L_1, B \in L_2$. Moreover, if $X \subseteq H(B)$ or $B \subseteq L(X)$, the necessary, sufficient granule concept in Proposition 8 degenerates to the sufficient and necessary granule concepts in Proposition 1; the sufficient and necessary granule concept in Propositions 6–7 degenerates to the sufficient and necessary granule concept of Proposition 1.

So far, a movement viewpoint of the TCCL method is completed, and it is obvious that all the granule concepts can be found after movement, where the semantic interpretation of all the movements of the concept is considered in this section. Next, we will mainly introduce the two-way granule concept space to complete the cognitive process of TCCL.

B. Concept Movement

According to the above discussion, we can obtain the method of granule concept learning in two ways. One is learning concepts from the top to bottom: the theories of evolving arbitrary information granules, sufficient granule concepts, and necessary granule concepts into sufficient and necessary granule concepts. The other method is from bottom to top: evolving sufficient and necessary granule concepts into sufficient or necessary granule concepts. However, we still do not know how to initial granule concept space from a movement perspective. Hence, we will discuss the concept movement mechanism of the granule concept space in TCCL.

Definition 2: Let (L_1, L_2, L, H) be a cognitive system, where L and H be a pair of cognitive operators. A weak tri-partition $\mathcal{G}$ is a set of three subspaces, denoted by $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1 \cap \mathcal{G}_2\}$. For $(X^L, B^H) \in \mathcal{G}$, we have

$$\begin{aligned} \mathcal{G}'_1 &= \{(X^L, B^H) | \mathcal{G}^p((X^L, B^H)) \rightsquigarrow_{LH} \mathcal{G}^1\} \\ \mathcal{G}'_2 &= \{(X^L, B^H) | \mathcal{G}^p((X^L, B^H)) \rightsquigarrow_{LH} \mathcal{G}^2\} \\ \mathcal{G}'_1 \cap \mathcal{G}'_2 &= \{(X^L, B^H) | \mathcal{G}^p((X^L, B^H)) \rightsquigarrow_{LH} \mathcal{G}^3\} \end{aligned} \tag{7}$$

where $\mathcal{G}'_1$, $\mathcal{G}'_2$, $\mathcal{G}'_1 \cap \mathcal{G}'_2$ are the new two-way granule concept spaces, respectively.

It should be pointed out that (X^L, B^H) is the two-way granule concept obtained through cognitive learning in the cognitive system (L_1, L_2, L, H), where $X \in L_1$ and $B \in L_2$.

Corollary 2: Let (L_1, L_2, L, H) be a cognitive system, where L and H be a pair of cognitive operators. For $\mathcal{G}'_1 \cup \mathcal{G}'_2$ is a finite nonempty granule concept space. A weak tri-partition $\mathcal{G}' = \{\mathcal{G}'_1, \mathcal{G}'_2, \mathcal{G}'_1 \cap \mathcal{G}'_2\}$. For $(X^L, B^H) \in \mathcal{G}'_1 \cup \mathcal{G}'_2$, we have

$$\begin{aligned} \mathcal{G}'_1 &= \{(X^L, B^H) | B^H \leqslant L(X^L), X^L \leqslant H(B^H)\} \\ \mathcal{G}'_2 &= \{(X^L, B^H) | L(X^L) \leqslant B^H, H(B^H) \leqslant X^L\} \\ \mathcal{G}'_1 \cap \mathcal{G}'_2 &= \{(X^L, B^H) | B^H = L(X^L), X^L = H(B^H)\}. \end{aligned} \tag{8}$$

According to the above analysis, one can learn two-way granule concepts from arbitrary information granules or granule concepts through the learning mechanism of TCCL. Meanwhile, the concept movement strategy of TCCL is described in Algorithm 3.

Algorithm 3 Process of Concept Movement

Input: Arbitrary information granule (X, B), a dataset G.
Output: Necessary, sufficient, sufficient, and necessary granule concept space: $\mathcal{G}'_1$, $\mathcal{G}'_2$, $\mathcal{G}'_3$.
1: Construct the necessary, sufficient, sufficient, and necessary granule concept space: $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ by Alg.1;
2: **for** $(X^L, B^H) \in \mathcal{G}_3$ **do**
3: Learn necessary granule concept granule concept (X^1, B^1), $(X^1{}_1, B^1{}_1), (X^1{}_2, B^1{}_2), \ldots, (X^1{}_m, B^1{}_m)$, $m \leqslant 6$ from (X^L, B^H), according to Proposition 6;
4: Learn sufficient granule concept granule concept (X^2, B^2), $(X^2{}_1, B^2{}_1), (X^2{}_2, B^2{}_2), \ldots, (X^2{}_m, B^2{}_m)$, $m \leqslant 6$ from (X^L, B^H), according to Proposition 7;
5: Learn sufficient and necessary granule concept (X^3, B^3), $(X^3{}_1, B^3{}_1), (X^3{}_2, B^3{}_2), \ldots, (X^3{}_m, B^3{}_m)$, $m \leqslant 6$ according to Proposition 8;
6: **end for**
7: $\mathcal{G}'_1 \leftarrow \mathcal{G}_1 \cup (X^1, B^1)$; $\mathcal{G}'_2 \leftarrow \mathcal{G}_2 \cup (X^2, B^2)$; $\mathcal{G}'_3 \leftarrow \mathcal{G}_3 \cup (X^3, B^3)$.

Now, we can analyze the time complexity of Algorithm 3. For step 1, it will call Algorithm 1. Thus, the time complexity of step 1 is $O(|U| \cdot (|X| + |AT|))$. In step 2, its running time is decided by the for-loop. Running steps 2–6 take $O(|U| \cdot (|X| + |AT|))$. Thus, the time complexity of Algorithm 3 takes $O(|U| \cdot (|X| + |AT|))$.

C. Concept Evolution of TCCL

According to the concept learning method in literature [31], we can learn a granule concept space $\mathcal{G}_{LH}$ from a formal context. Essentially, the granule concept in granule concept space is a sufficient and necessary granule concept. In this section, we study the concept evolution method of TCCL from a given formal context rather than clues.

We demonstrate that the two-way granule concept can evolve from a sufficient and necessary granule concept. Of course, we can still learn the corresponding two-way granule concepts from a formal context $F = (U, A, I)$. The method to learn the two-way granule concept from a formal context is as follows.

Case 4: The method to learn the necessary granule concept from a formal context in TCCL.

Proposition 9: Let $F = (U, A, I)$ be a formal context, L and H be two cognitive operators, and $\mathcal{G}_1$ be a necessary granule concept space in TCCL. If $(HL(x), L(x))$ and $(H(b), LH(b))$ be two granule concepts in the concept cognitive learning model (CCLM), as follows.

1) $(HL(x), L(x \vee H(b))) \in \mathcal{G}_1$.
2) $(HL(x \wedge H(b)), L(x)) \in \mathcal{G}_1$.
3) $(HL(x \wedge H(b)), L(x \vee H(b)) \in \mathcal{G}_1$.
4) $(H(b \vee L(x)), LH(b)) \in \mathcal{G}_1$.
5) $(H(b \vee L(x)), LH(b \wedge L(x))) \in \mathcal{G}_1$.
6) $(H(b), LH(b \wedge L(x))) \in \mathcal{G}_1$.

Proof. It is directly obtained from Definition 1 and Proposition 3.

Case 5: The method to learn the sufficient granule concept from a formal context in TCCL.

Proposition 10: Let $F = (U, A, I)$ be a formal context, L and H be two cognitive operators, and $\mathcal{G}_2$ be a sufficient granule concept space in TCCL. If $(HL(x), L(x))$ and $(H(b), LH(b))$ be two granule concepts in CCLM, as follows.

1) $(HL(x \vee H(b)), L(x)) \in \mathcal{G}_2$.
2) $(HL(x \vee H(b)), L(x \wedge H(b))) \in \mathcal{G}_2$.
3) $(HL(x), L(x \wedge H(b))) \in \mathcal{G}_2$.
4) $(H(b \wedge L(x)), LH(b)) \in \mathcal{G}_2$.
5) $(H(b \wedge L(x)), LH(b \vee L(x))) \in \mathcal{G}_2$.
6) $(H(b), LH(b \vee L(x))) \in \mathcal{G}_2$.

Proof: It is directly obtained from Definition 1 and Proposition 4.

Case 6: The method to learn the sufficient and necessary granule concept from a formal context in TCCL.

Proposition 11: Let $F = (U, A, I)$ be a formal context, L and H be two cognitive operators, and $\mathcal{G}_2$ be a sufficient granule concept space in TCCL. If $(HL(x), L(x))$ and $(H(b), LH(b))$ be two granule concepts in CCLM, as follows.

1) $(HL(x), L(x)) \in \mathcal{G}_1 \cap \mathcal{G}_2$.
2) $(H(b), LH(b)) \in \mathcal{G}_1 \cap \mathcal{G}_2$.
3) $(HL(x \wedge H(b)), L(x \wedge H(b))) \in \mathcal{G}_1 \cap \mathcal{G}_2$.
4) $(HL(x \vee H(b)), L(x \vee H(b))) \in \mathcal{G}_1 \cap \mathcal{G}_2$.
5) $(H(b \vee L(x)), LH(b \vee L(x))) \in \mathcal{G}_1 \cap \mathcal{G}_2$.
6) $(H(b \wedge L(x)), LH(b \wedge L(x))) \in \mathcal{G}_1 \cap \mathcal{G}_2$.

Proof: It is directly obtained from Definition 1 and Proposition 5.

Algorithm 4 Concept Evolution of TCCL Based on CCLM

Input: A formal context $F = (U, A, I)$

Output: Two-way granule concept space $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$.

1: Construct a sufficient and necessary granule concept space $\mathcal{G}_{LH}$ according to CCLM in paper [31];
2: **for all** $(HL(x), L(x)), (H(b), LH(b)) \in \mathcal{G}_{LH}$ **do**
3: **if** $HL(x) \not\subset H(b) \& H(b) \not\subset HL(x)$ **then**
4: Learn necessary granule concept (X^1, B^1) from $\mathcal{G}_{LH}$, $({X^1}_1, {B^1}_1), ({X^1}_2, {B^1}_2), \ldots, ({X^1}_m, {B^1}_m)$, $m \leqslant 6$ by six methods according to Proposition 9;
5: Learn sufficient granule concept (X^2, B^2) from $\mathcal{G}_{LH}$, $({X^2}_1, {B^2}_1), ({X^2}_2, {B^2}_2), \ldots, ({X^2}_m, {B^2}_m)$, $m \leqslant 6$ by six methods according to Proposition 10;
6: Learn sufficient and necessary granule concept (X^3, B^3) from $\mathcal{G}_{LH}$, $({X^3}_1, {B^3}_1), \ldots, ({X^3}_m, {B^3}_m)$, $m \leqslant 6$ by six methods according to Proposition 11;
7: **end if**
8: **end for**
9: $\mathcal{G}_1 \leftarrow (X^1, B^1)$; $\mathcal{G}_2 \leftarrow (X^2, B^2)$; $\mathcal{G}_3 \leftarrow (X^3, B^3)$.

According to the above discussion, TCCL can evolute concepts based on granule concepts in CCLM from a formal context. Meanwhile, the detail of TCCL is described in Algorithm 4. Now, we can analyze the time complexity of Algorithm 4. For step 1, it will call Algorithm 1, the running time complexity is $O(|U| \cdot |AT|))$. Thus, the time complexity of step 2 is $O(|\mathcal{G}_{LH}|)$. Running steps 3–7 take $O(|U|^2 \cdot |AT|)$. Thus, running time complexity of Algorithm 4 takes $O(|U|^2 \cdot |AT| \cdot |\mathcal{G}_{LH}|)$.

TABLE I
DATASET INFORMATION AND ITS FORMAL CONTEXT

No.s	Name	Sample	Feature	Class	Pre-processing
1	Zoo	101	17	7	/
2	Breast Cancer	286	9	2	/
3	Monk-2	432	8	2	/
4	Appendicitis	106	7	2	Discretization
5	Glass Identification	214	9	6	Discretization
6	Tic-Tac-Toe	958	9	2	/
7	Mammographic Mass	961	5	2	Discretization
8	Vowel	990	10	11	Discretization
9	Banknote	1372	4	2	Discretization
10	Mushroom	8124	22	2	/
11	HTRU-2	17898	8	2	Discretization
12	Skin Segmentation	245057	3	2	Discretization

V. EXAMPLE ANALYSIS

A formal context of the situations of developing countries is presented in Appendix I. This formal context is denoted by $F = (U, A, I)$. There are 128 objects that represent the kinds of developing countries. The data are from [2] and [26]. The descriptions of the characteristics including: a_1: Group of 77; a_2: nonaligned; a_3: least less developed country (LLDC); a_4: most seriously affected country (MSAC); a_5: organization of petroleum exporting countries (OPEC); a_6: African, Caribbean, and Pacific associables (ACP).

When the United Nations plan to grant loans to developing countries to support them with economic growth. The United Nations need fully acknowledge the political and economic environment for the distribution of equity. The United Nations must take attribute set A into account when decision-making about which countries to choose. The approach introduced in the article can be used to choose the candidate countries.

It is supposed that (X_0, B_0) is an arbitrary information granule in a formal context, where X_0 is the countries and B_0 is the characteristics. However, (X_0, B_0) is just a clue given at the beginning, and it induces the condition that the countries selected do not satisfy the given characteristics and the countries satisfying the given characteristics are not selected. Then, given $X_0 = \{x_1, x_{11}, x_{14}, x_{35}, x_{47}, x_{52}, x_{59}, x_{78}, x_{84}, x_{92}, x_{87}, x_{95}, x_{106}\}$ as well as $B_0 = \{a_1, a_2, a_3, a_6\}$.

When the funding is controlled, the United Nations can only choose the countries which must meet the given condition characteristic. Thus, the necessary information granule concept is a great selection. Note that (X_0, B_0) satisfies $X_0 \leqslant L(B_0)$. Therefore, the countries in X_0 satisfy and precede the given characteristic in B_0.

If the United Nations decides to consider as many countries as possible, they may relax the conditions of some developing countries. Note that a sufficient granule concept is a great choice. Given (X_0, B_0) be the initial information granule, and then we can obtain the sufficient granule concept of (X_0, B_0) by Alg.1: $(X_1, B_1), (X_2, B_2) \in \mathcal{G}_2$, where $X_1 = \{x_1, x_{11}, x_{14}, x_{17}, x_{18}, x_{21}, x_{22}, x_{23}, x_{27}, x_{31}, x_{35}, x_{37}, x_{38}, x_{41}, x_{42}, x_{45}, x_{46}, x_{47}, x_{52}, x_{59}, x_{64}, x_{67}, x_{68}, x_{71}, x_{72}, x_{78}, x_{83}, x_{84}, x_{92}, x_{94}, x_{95}, x_{96}, x_{100}, x_{103}, x_{106}, x_{108}, x_{112}, x_{114}, x_{119}, x_{122}, x_{126}, x_{127}\}$, $X_2 = U - \{x_{16}, x_{25}, x_{58}, x_{75}, x_{80}, x_{93}, x_{105}, x_{118}\}$.$B_1 = \{a_1, a_2, a_3, a_6\}$, $B_2 = \{a_1, a_2, a_3, a_6\}$.

TABLE II

NUMBER OF TWO-WAY GRANULE CONCEPT (AVE.±SD) IN ZOO

Objects		2%				5%				10%				15%			
Attribute		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$
15%	Alg.1	4.5±0.3	2.0±0.0	2.0±0.0	8.5±0.3	5.2±0.1	2.0±0.0	1.9±0.3	9.1±0.4	4.0±0.1	2.0±0.0	1.2±0.4	7.2±0.5	3.5±1.0	2.0±0.0	1.1±0.3	6.6±1.3
	Alg.2	3.8±1.0	3.4±0.0	1.6±0.7	8.8±1.7	3.9±0.1	3.4±0.0	1.4±0.5	8.7±0.6	2.4±0.9	2.3±0.0	1.1±0.3	5.8±1.2	2.1±0.3	2.1±0.0	1.1±0.3	5.3±0.6
	Alg.3	**5.7±0.6**	**3.5±0.9**	**2.4±0.7**	**11.6±2.2**	**6.8±0.9**	**4.7±0.5**	**2.7±0.8**	**14.2±2.2**	**5.6±0.7**	**4.1±0.8**	**2.0±0.9**	**11.7±2.5**	**4.3±0.6**	**3.6±0.9**	**1.5±0.7**	**9.4±2.2**
	2WL	4.5±0.3	2.0±0.0	**2.4±0.7**	8.9±1.0	5.2±0.1	2.0±0.0	**2.7±0.8**	9.9±0.9	4.0±0.1	2.0±0.0	**2.0±0.9**	8.0±1.0	3.5±0.9	2.0±0.0	**1.5±0.7**	7.0±1.6
20%	Alg.1	5.4±1.0	2.0±0.0	1.8±0.4	9.2±1.4	5.5±0.8	2.0±0.0	1.8±0.4	9.3±1.2	4.4±0.9	2.0±0.0	1.4±0.5	7.8±1.4	3.9±0.3	2.0±0.0	1.0±0.0	6.9±0.3
	Alg.2	4.2±0.2	3.5±0.0	1.2±0.6	8.9±0.8	4.2±0.2	3.5±0.0	0.9±0.5	8.6±0.7	2.8±0.2	2.6±0.0	1.2±0.4	6.6±0.6	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0
	Alg.3	**7.3±0.7**	**5.0±0.3**	**3.0±0.1**	**15.3±1.1**	**7.6±0.4**	**5.0±0.6**	**2.9±0.0**	**15.5±1.0**	**6.2±0.5**	**4.6±0.8**	**2.3±0.0**	**13.1±1.3**	**5.0±0.1**	**3.6±0.5**	**1.5±0.5**	**10.1±1.1**
	2WL	5.4±0.9	2.0±0.0	**3.0±0.1**	10.4±1.0	5.5±0.8	2.0±0.0	**2.9±0.0**	10.4±0.8	4.4±0.9	2.0±0.0	**2.3±0.0**	8.7±0.9	3.9±0.3	2.0±0.0	**1.5±0.5**	7.4±0.8
30%	Alg.1	4.7±0.1	2.0±0.0	1.4±0.5	8.1±0.6	3.7±0.6	2.0±0.0	1.0±0.4	6.7±1.0	3.0±0.3	2.0±0.0	0.5±0.7	5.5±1.0	3.4±0.3	2.0±0.0	0.7±0.6	6.1±0.9
	Alg.2	3.2±0.5	2.8±0.0	1.1±0.3	7.1±0.8	2.0±0.4	2.0±0.0	1.0±0.4	5.0±0.8	1.7±0.2	1.6±0.0	0.5±0.7	3.8±0.9	1.7±0.6	1.7±0.0	0.7±0.6	4.1±1.2
	Alg.3	**7.0±0.7**	**4.6±0.8**	**2.7±0.9**	**14.3±2.4**	**4.2±0.9**	**3.7±0.6**	**1.2±0.6**	**9.1±2.1**	**3.5±2.2**	**3.0±0.3**	**0.7±0.2**	**7.2±2.7**	**3.6±0.5**	**3.2±0.1**	**0.8±0.7**	**7.6±1.3**
	2WL	4.7±0.1	2.0±0.0	**2.7±0.9**	9.4±1.0	3.7±0.6	2.0±0.0	**1.2±0.6**	6.9±1.2	3.0±0.3	2.0±0.0	**0.7±0.2**	5.7±0.5	3.4±0.3	2.0±0.0	**0.8±0.7**	6.2±1.0

TABLE III

NUMBER OF TWO-WAY GRANULE CONCEPT (AVE.±SD) IN BREAST CANCER

Objects		2%				5%				10%				15%			
Attribute		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$
15%	Alg.1	4.0±0.0	2.0±0.0	1.0±0.0	7.0±0.0	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0
	Alg.2	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0
	Alg.3	**4.3±0.5**	**4.0±0.0**	**1.4±0.5**	**9.7±1.0**	**4.0±0.0**	**4.0±0.0**	**1.2±0.4**	**9.2±0.4**	**4.0±0.0**	**4.0±0.0**	**1.1±0.3**	**9.1±0.3**	**4.0±0.0**	**4.0±0.0**	**1.5±0.5**	**9.5±0.5**
	2WL	4.0±0.0	2.0±0.0	**1.4±0.5**	7.4±0.5	**4.0±0.0**	2.0±0.0	**1.2±0.4**	7.2±0.4	**4.0±0.0**	2.0±0.0	**1.1±0.3**	7.1±0.3	**4.0±0.0**	2.0±0.0	**1.5±0.5**	7.5±0.5
20%	Alg.1	4.0±0.0	2.0±0.0	1.0±0.0	7.0±0.0	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0
	Alg.2	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0
	Alg.3	**4.1±0.3**	**4.0±0.0**	**1.6±0.5**	**9.7±0.8**	**4.0±0.0**	**4.0±0.0**	**1.4±0.5**	**9.4±0.5**	**4.0±0.0**	**4.0±0.0**	**1.4±0.5**	**9.4±0.5**	**4.0±0.0**	**4.0±0.0**	**1.4±0.5**	**9.4±0.5**
	2WL	4.0±0.0	2.0±0.0	**1.6±0.5**	7.6±0.5	**4.0±0.0**	2.0±0.0	**1.4±0.5**	7.4±0.5	**4.0±0.0**	2.0±0.0	**1.4±0.5**	7.4±0.5	**4.0±0.0**	2.0±0.0	**1.4±0.5**	7.4±0.5
30%	Alg.1	4.0±0.0	2.0±0.0	1.0±0.0	7.0±0.0	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0	**4.0±0.0**	2.0±0.0	**1.0±0.0**	7.0±0.0	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0
	Alg.2	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	**1.0±0.0**	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0
	Alg.3	**4.3±0.5**	**4.0±0.0**	**1.9±0.3**	**10.2±0.8**	**4.0±0.0**	**4.0±0.0**	**1.4±0.5**	**9.4±0.5**	**4.0±0.0**	**4.0±0.0**	**1.0±0.0**	**9.0±0.0**	**4.0±0.0**	**4.0±0.0**	**1.5±0.5**	**9.5±0.5**
	2WL	4.0±0.0	2.0±0.0	**1.9±0.3**	7.9±0.3	**4.0±0.0**	2.0±0.0	**1.4±0.5**	7.4±0.5	**4.0±0.0**	2.0±0.0	**1.0±0.0**	7.0±0.0	**4.0±0.0**	2.0±0.0	**1.5±0.5**	7.5±0.5

TABLE IV

NUMBER OF TWO-WAY GRANULE CONCEPT (AVE.±SD) IN MONKS-2

Objects		2%				5%				10%				15%			
Attribute		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$
15%	Alg.1	5.6±0.2	1.9±0.3	1.9±0.3	9.4±0.8	5.6±0.2	1.9±0.3	1.9±0.3	9.4±0.8	5.6±0.2	1.9±0.3	1.9±0.3	9.4±0.8	6.0±0.0	2.0±0.0	2.0±0.0	10.0±0.0
	Alg.2	4.6±0.2	3.7±0.3	0.9±0.3	9.2±0.8	4.6±0.2	3.7±0.3	0.9±0.3	9.2±0.8	4.6±0.2	3.7±0.3	1.0±0.0	9.3±0.5	5.0±0.0	4.0±0.0	0.9±0.3	9.9±0.3
	Alg.3	**7.7±2.0**	**4.5±0.2**	**3.4±0.0**	**15.6±2.2**	**7.3±2.0**	**4.2±0.2**	**3.1±0.9**	**14.6±3.1**	**6.7±0.7**	**3.8±1.0**	**2.9±0.7**	**13.4±2.4**	**7.0±0.0**	**4.1±0.3**	**2.9±0.3**	**14.0±0.6**
	2WL	5.6±0.2	1.9±0.3	**3.4±0.0**	10.9±0.5	5.6±0.2	1.9±0.3	**3.1±0.9**	10.6±1.4	5.6±0.2	1.9±0.3	**2.9±0.7**	10.4±1.2	6.0±0.0	2.0±0.0	**2.9±0.3**	10.9±0.3
20%	Alg.1	5.6±0.8	2.0±0.0	1.9±0.3	9.5±1.1	5.8±0.6	2.0±0.0	2.0±0.0	9.8±0.6	5.6±0.8	2.0±0.0	1.9±0.3	9.5±1.1	5.2±1.0	2.0±0.0	1.6±0.5	8.8±1.5
	Alg.2	4.5±0.0	3.7±0.0	0.9±0.5	9.1±0.5	4.8±0.6	3.9±0.0	0.8±0.6	8.7±1.2	4.5±0.0	3.7±0.0	0.9±0.5	9.1±0.5	3.8±0.5	3.2±0.0	0.8±0.4	7.8±0.9
	Alg.3	**7.0±0.2**	**4.8±0.6**	**2.8±0.9**	**14.6±1.7**	**7.2±0.3**	**4.6±0.5**	**2.9±0.8**	**14.7±1.6**	**6.8±0.6**	**4.4±0.5**	**2.7±0.9**	**13.9±2.0**	**6.0±0.7**	**4.3±0.5**	**4.3±0.5**	**14.6±1.7**
	2WL	5.6±0.8	2.0±0.0	**2.8±0.9**	10.4±1.7	5.8±0.6	2.0±0.0	**2.9±0.8**	10.7±1.4	5.6±0.8	2.0±0.0	**2.7±0.9**	10.3±1.7	5.2±1.0	2.0±0.0	2.1±0.0	9.3±1.0
30%	Alg.1	4.6±0.9	2.0±0.0	1.6±0.5	8.2±1.4	4.0±0.8	2.0±0.0	1.3±0.5	7.3±1.3	4.5±0.0	2.0±0.0	1.4±0.5	7.9±0.5	2.1±0.0	2.0±0.0	1.4±0.5	5.5±0.5
	Alg.2	3.2±0.2	2.9±0.0	1.2±0.6	7.3±0.8	2.5±0.9	2.4±0.0	1.1±0.5	6.0±1.4	3.0±0.3	2.7±0.0	0.9±0.5	6.6±0.8	2.8±0.2	2.6±0.0	1.1±0.5	6.5±0.7
	Alg.3	**5.4±0.6**	**4.6±0.5**	**2.0±0.9**	**12.0±2.0**	**4.2±0.1**	**4.0±0.6**	**1.3±0.5**	**9.5±1.2**	**4.9±0.6**	**4.2±0.6**	**1.6±0.9**	**10.7±2.1**	**4.7±0.5**	**4.2±0.6**	**1.6 ±0.8**	**10.5±1.9**
	2WL	4.6±0.9	2.0±0.0	**2.0±0.9**	8.6±1.8	4.0±0.8	2.0±0.0	**1.3±0.5**	7.3±1.3	4.5±0.0	2.0±0.0	**1.6±0.9**	8.1±0.9	4.3±0.9	2.0±0.0	**1.6 ±0.8**	7.9±1.7

Meanwhile, we can obtain the sufficient granule concept of (X_0, B_0) by Algorithm 2: $(X_2, B_2), (X_3, B_3), (X_4, B_4) \in \mathcal{G}_2$, where $X_3 = \{x_{11}, x_{14}, x_{17}, x_{18}, x_{21}, x_{22}, x_{23}, x_{27}, x_{31}, x_{37}, x_{38}, x_{41}, x_{42}, x_{45}, x_{46}, x_{64}, x_{67}, x_{68}, x_{71}, x_{72}, x_{83}, x_{84}, x_{94}, x_{96}, x_{100}, x_{103}, x_{108}, x_{112}, x_{114}, x_{119}, x_{122}, x_{126}, x_{127}\}$, $X_4 = U - \{x_{16}, x_{25}, x_{58}, x_{75}, x_{80}, x_{93}, x_{105}, x_{118}\}.B_3 = \{a_1, a_2, a_3, a_6\}, B_4 = \{a_1\}$.

Moreover, we can obtain the sufficient granule concept of (X_0, B_0) by Algorithm 3: $(X_1, B_1) \in \mathcal{G}_2$, $(X_2, B_2) \in \mathcal{G}_2$, $(X_3, B_3) \in \mathcal{G}_2$ and $(X_4, B_4) \in \mathcal{G}_2$.

Note that the sufficient granule concepts (X_1, B_1), (X_2, B_2), and (X_3, B_3) say that the countries X_1, X_2, and X_3 only meet or do not meet the characteristic B_1, respectively. While the sufficient granule concepts (X_4, B_4) only meet or do not meet the characteristic B_4, this condition is relatively lenient.

If the United Nations hopes that the selected countries must satisfy the given attributes, all the countries that meet the shared characteristics must be chosen. Now the sufficient and necessary granule concepts are a good choice. Thus, we can compute the sufficient and necessary granule concepts of (X_0, B_0) by Algorithm 1: $(X_5, B_5) \in \mathcal{G}_1 \cap \mathcal{G}_2$ and $(X_6, B_6) \in \mathcal{G}_1 \cap \mathcal{G}_2$. Meanwhile, we can obtain the sufficient and necessary granule concept of (X_0, B_0) by Algorithm 2 is the same as Algorithm 1, where $X_5 = \{x_{11}, x_{14}, x_{17}, x_{18}, x_{21}, x_{22}, x_{23}, x_{27}, x_{31}, x_{37}, x_{38}, x_{41}, x_{42}, x_{45}, x_{46}, x_{64}, x_{67}, x_{68}, x_{71}, x_{72}, x_{83}, x_{94}, x_{96}, x_{100}, x_{103}, x_{108}, x_{112}, x_{114}, x_{119}, x_{122}, x_{126}, x_{127}\}$, $X_6 = U - \{x_{16}, x_{25}, x_{58}, x_{75}, x_{80}, x_{93}, x_{105}, x_{118}\}$. $B_5 = \{a_1, a_2, a_3, a_6\}$, $B_6 = \{a_1\}$.

Meanwhile, we can obtain the sufficient and necessary granule concepts of (X_0, B_0) by Algorithm 3 is the same as Algorithm 1, that is, (X_5, B_5) and (X_6, B_6). The sufficient and necessary granule concepts (X_5, B_5) and (X_6, B_6) say that the countries X_5 and X_6 meet the characteristics B_5 and B_6, respectively.

Moreover, the outcome of Algorithm 1 in this article is the same in [26]. Besides, we can learn more granule concepts via Algorithms 2 and 3 than the method in [26]. For Algorithm 4, we will verify its validity in Section VI-C.

VI. EXPERIMENTS

In this section, we validate the effectiveness of TCCL for the performance of concept learning based on a formal context. Specifically, we compare TCCL with other CCL mechanisms, including TWC [22], 2WL [26], and CCLM [31]. All experimental setup of comparison methods is consistent with corresponding references. The experimental computing

TABLE V

NUMBER OF TWO-WAY GRANULE CONCEPT (AVE.±SD) IN APPENDICITIS

Objects		2%				5%				10%				15%			
Attribute		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$
15%	Alg.1	2.0±0.0	1.0±0.0	**1.0±0.0**	4.0±0.0	**1.6±0.5**	0.6±0.5	**0.6±0.5**	2.8±1.5	1.1±0.3	0.1±0.3	**0.1±0.3**	1.3±0.9	**1.0±0.0**	**0.0±0.0**	**0.0±0.0**	**1.0±0.0**
	Alg.2	2.0±0.0	1.0±0.0	**1.0±0.0**	4.0±0.0	1.2±1.0	0.6±0.5	**0.6±0.5**	2.4±2.0	0.2±0.6	0.1±0.3	**0.1±0.3**	0.4±1.2	0.0±0.0	**0.0±0.0**	**0.0±0.0**	0.0±0.0
	Alg.3	**2.8±0.4**	**2.0±0.0**	**1.0±0.0**	**5.8±0.4**	**1.6±0.5**	**1.2±0.5**	**0.6±0.5**	**3.4±1.5**	**1.1±0.3**	**0.2±0.3**	**0.1±0.3**	**1.4±0.9**	**1.0±0.0**	**0.0±0.0**	**0.0±0.0**	**1.0±0.0**
	2WL	2.0±0.0	1.0±0.0	**1.0±0.0**	4.0±0.0	**1.6±0.5**	0.6±0.5	**0.6±0.5**	2.8±1.5	1.1±0.3	0.1±0.3	**0.1±0.3**	1.3±0.9	**1.0±0.0**	**0.0±0.0**	**0.0±0.0**	**1.0±0.0**
20%	Alg.1	2.0±0.0	1.0±0.0	**1.0±0.0**	4.0±0.0	**1.2±0.4**	0.2±0.4	**0.2±0.4**	1.6±1.2	1.0±0.0	0.0±0.0	**0.0±0.0**	**1.0±0.0**	**1.0±0.0**	**0.0±0.0**	**0.0±0.0**	**1.0±0.0**
	Alg.2	2.0±0.0	1.0±0.0	**1.0±0.0**	4.0±0.0	0.4±0.8	0.2±0.4	**0.2±0.4**	1.0±1.6	0.0±0.0	0.0±0.0	**0.0±0.0**	0.0±0.0	0.0±0.0	**0.0±0.0**	**0.0±0.0**	0.0±0.0
	Alg.3	**2.4±0.5**	**2.0±0.0**	**1.0±0.0**	**5.4±0.5**	**1.2±0.4**	**0.4±0.4**	**0.2±0.4**	**1.8±1.2**	**1.0±0.0**	**0.0±0.0**	**0.0±0.0**	**1.0±0.0**	**1.0±0.0**	**0.0±0.0**	**0.0±0.0**	**1.0±0.0**
	2WL	2.0±0.0	1.0±0.0	**1.0±0.0**	4.0±0.0	**1.2±0.4**	0.2±0.4	**0.2±0.4**	1.6±1.2	**1.0±0.0**	**0.0±0.0**	**0.0±0.0**	**1.0±0.0**	**1.0±0.0**	**0.0±0.0**	**0.0±0.0**	**1.0±0.0**
30%	Alg.1	5.0±0.3	2.0±0.0	1.9±0.3	8.9±0.6	4.2±0.6	1.9±0.3	1.3±0.6	7.4±1.5	3.3±0.5	2.0±0.0	1.0±0.0	6.3±0.5	3.3±0.6	2.0±0.0	0.9±0.3	6.2±0.9
	Alg.2	4.1±0.1	3.5±0.0	1.4±0.4	9.0±0.5	3.0±0.7	2.6±0.3	0.9±0.3	6.5±1.3	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	1.9±0.3	1.9±0.0	0.9±0.3	4.7±0.6
	Alg.3	**6.5±2.1**	**4.5±0.7**	**2.6±0.8**	**13.6±3.6**	**5.9±2.6**	**3.9±0.3**	**2.3±0.3**	**12.1±3.2**	**4.5±0.1**	**3.6±0.5**	**1.6±0.5**	**9.7±1.1**	**3.9±0.1**	**3.2±0.6**	**1.2±0.6**	**8.3±1.3**
	2WL	5.0±0.3	2.0±0.0	2.6±0.8	9.6±1.1	4.2±0.6	1.9±0.3	**2.3±0.3**	8.4±1.2	3.3±0.5	2.0±0.0	1.6±0.5	6.9±1.0	3.3±0.6	2.0±0.0	1.2±0.6	6.5±1.2

TABLE VI

NUMBER OF TWO-WAY GRANULE CONCEPT (AVE.±SD) IN GLASS IDENTIFICATION

Objects		2%				5%				10%				15%			
Attribute		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$
15%	Alg.1	4.0±0.0	2.0±0.0	1.0±0.0	7.0±0.0	4.1±0.3	2.0±0.0	1.1±0.3	7.2±0.6	4.4±0.8	2.0±0.0	1.2±0.4	7.6±1.2	4.2±0.6	2.0±0.0	**1.1±0.3**	7.3±0.9
	Alg.2	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.1±0.3	2.1±0.0	1.1±0.3	5.3±0.6	2.6±0.2	2.4±0.0	1.0±0.0	6.0±0.2	2.3±0.9	2.2±0.0	0.9±0.3	5.4±1.2
	Alg.3	**4.8±0.6**	**4.0±0.0**	**1.5±0.5**	**10.3±1.1**	**4.5±0.5**	**3.9±0.3**	**1.2±0.4**	**9.6±1.2**	**4.7±0.4**	**4.4±0.8**	**1.6±0.8**	**10.7±2.0**	**4.3±0.9**	**3.9±0.5**	**1.1±0.3**	**9.3±1.7**
	2WL	4.0±0.0	2.0±0.0	**1.5±0.5**	7.5±0.5	4.1±0.3	2.0±0.0	**1.2±0.4**	7.3±0.7	4.4±0.8	2.0±0.0	**1.6±0.8**	8.0±1.6	4.2±0.6	2.0±0.0	**1.1±0.3**	7.3±0.9
20%	Alg.1	4.3±0.6	2.0±0.0	1.2±0.4	7.5±1.0	4.0±0.0	2.0±0.0	1.0±0.0	7.0±0.0	4.4±0.8	2.0±0.0	1.2±0.4	7.6±1.2	4.2±0.6	2.0±0.0	**1.1±0.3**	7.3±0.9
	Alg.2	2.2±0.4	2.2±0.0	1.2±0.4	5.6±0.8	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.4±0.9	2.3±0.0	1.1±0.3	7.8±1.2	2.3±0.9	2.2±0.0	1.0±0.0	5.5±0.9
	Alg.3	**5.4±0.4**	**4.2±1.0**	**1.6±0.8**	**11.2±2.2**	**4.5±0.5**	**3.8±0.4**	**1.2±0.4**	**9.5±1.3**	**4.5±0.0**	**4.1±0.7**	**1.4±0.7**	**10±1.4**	**4.3±0.9**	**3.8±0.6**	**1.5±0.7**	**9.6±2.2**
	2WL	4.3±0.6	2.0±0.0	**1.6±0.8**	7.9±0.3	4.0±0.0	2.0±0.0	**1.2±0.4**	7.2±0.4	4.4±0.8	2.0±0.0	**1.4±0.7**	7.8±1.5	4.2±0.6	2.0±0.0	**1.5±0.7**	7.7±1.3
30%	Alg.1	4.0±0.0	2.0±0.0	1.0±0.0	7.0±0.0	4.0±0.0	2.0±0.0	1.0±0.0	7.0±0.0	**4.0±0.0**	2.0±0.0	**1.0±0.0**	7.0±0.0	**3.9±0.3**	2.0±0.0	**1.0±0.0**	6.9±0.3
	Alg.2	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	**1.0±0.0**	5.0±0.0	2.0±0.0	2.0±0.0	**1.0±0.0**	5.0±0.0
	Alg.3	**5.1±0.5**	**3.7±0.5**	**1.6±0.5**	**10.4±1.5**	**4.3±0.5**	**3.5±0.5**	**1.3±0.5**	**9.1±1.5**	**4.0±0.0**	**3.4±0.5**	**1.0±0.0**	**8.4±0.5**	**3.9±0.3**	**3.5±0.5**	**1.0±0.0**	**8.4±0.8**
	2WL	4.0±0.0	2.0±0.0	**1.6±0.5**	7.6±0.5	4.0±0.0	2.0±0.0	**1.3±0.5**	7.3±0.5	**4.0±0.0**	2.0±0.0	**1.0±0.0**	7.0±0.0	**3.9±0.3**	2.0±0.0	**1.0±0.0**	6.9±0.3

TABLE VII

NUMBER OF TWO-WAY GRANULE CONCEPT (AVE.±SD) IN TIC-TAC-TOE

Objects		2%				5%				10%				15%			
Attribute		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$
15%	Alg.1	5.6±0.8	2.0±0.0	1.9±0.3	9.0±1.1	3.0±0.4	2.0±0.0	0.9±0.3	5.9±0.7	2.9±0.5	2.0±0.0	0.8±0.4	5.7±0.9	3.0±0.4	2.0±0.0	0.9±0.3	5.0±0.7
	Alg.2	4.5±1.0	3.7±0.0	0.9±0.5	9.1±1.5	1.9±0.3	1.9±0.0	0.9±0.3	4.7±0.6	1.8±0.4	1.8±0.0	0.8±0.4	4.4±0.8	1.9±0.3	1.9±0.0	0.9±0.3	4.7±0.6
	Alg.3	**7.1±1.6**	**4.9±0.5**	**2.7±0.9**	**14.7±3.0**	**3.8±0.9**	**3.6±0.7**	**1.5±0.7**	**8.9±2.3**	**3.7±1.1**	**3.4±0.8**	**1.0±0.6**	**8.1±2.5**	**3.8±1.2**	**3.4±0.7**	**1.3±0.6**	**8.5±2.5**
	2WL	5.6±0.8	2.0±0.0	**2.7±0.9**	10.3±1.7	3.0±0.4	2.0±0.0	**1.5±0.7**	6.5±1.1	2.9±0.5	2.0±0.0	**1.0±0.6**	5.9±1.1	3.0±0.4	2.0±0.0	**1.3±0.6**	6.3±1.0
20%	Alg.1	5.4±0.9	2.0±0.0	1.7±0.5	9.1±1.4	2.7±0.8	2.0±0.0	0.5±0.5	5.2±1.3	2.7±0.8	2.0±0.0	0.5±0.5	5.2±1.3	2.8±0.9	2.0±0.0	0.5±0.5	5.3±1.4
	Alg.2	4.1±1.4	3.4±0.0	0.7±0.5	8.2±1.9	1.5±0.5	1.5±0.0	0.5±0.5	3.5±1.0	1.5±0.5	1.5±0.0	0.5±0.5	3.5±1.0	1.5±0.5	1.5±0.0	0.5±0.5	3.5±1.0
	Alg.3	**6.4±1.6**	**4.6±0.5**	**2.2±1.0**	**13.2±3.1**	**3.2±1.2**	**3.0±1.0**	**0.6±0.7**	**6.8±2.9**	**3.3±1.3**	**3.0±1.0**	**0.8±0.9**	**7.1±3.2**	**3.4±1.4**	**3.0±1.0**	**0.7±0.8**	**7.1±3.2**
	2WL	5.4±0.9	2.0±0.0	**2.2±1.0**	9.6±1.9	2.7±0.8	2.0±0.0	**0.6±0.7**	5.3±1.5	2.7±0.8	2.0±0.0	**0.8±0.9**	5.5±1.7	2.8±0.9	2.0±0.0	**0.7±0.8**	5.5±1.7
30%	Alg.1	**3.8±0.4**	2.0±0.0	**1.0±0.0**	6.8±0.4	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
	Alg.2	2.0±0.0	2.0±0.0	**1.0±0.0**	5.0±0.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0
	Alg.3	**3.8±0.4**	**3.8±0.4**	**1.0±0.0**	**8.6±0.8**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
	2WL	**3.8±0.4**	2.0±0.0	**1.0±0.0**	6.8±0.4	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**

TABLE VIII

NUMBER OF TWO-WAY GRANULE CONCEPT (AVE.±SD) IN MAMMOGRAPHIC MASS

Objects		2%				5%				10%				15%			
Attribute		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$
15%	Alg.1	**4.0±0.0**	2.0±0.0	**1.0±0.0**	7.0±0.0	**2.8±1.0**	2.0±0.0	**3.3±0.9**	8.1±1.9	2.0±0.0	**0.7±0.5**	**2.8±1.0**	5.5±1.5	2.0±0.0	**3.3±0.9**	2.0±0.0	7.3±0.9
	Alg.2	2.0±0.0	2.0±0.0	**1.0±0.0**	5.0±0.0	1.4±0.5	1.4±0.0	1.7±0.5	4.5±1.0	1.7±0.0	**0.7±0.5**	1.4±0.5	3.8±1.0	1.4±0.0	1.7±0.5	1.7±0.0	4.8±0.5
	Alg.3	**4.0±0.0**	**4.0±0.0**	**1.0±0.0**	**9.0±0.0**	**2.8±1.0**	**2.8±1.0**	**3.3±0.9**	**8.9±2.9**	**3.3±0.9**	**0.7±0.5**	**2.8±1.0**	**6.8±2.4**	**2.8±1.0**	**3.3±0.9**	**3.3±0.9**	**9.4±2.8**
	2WL	**4.0±0.0**	2.0±0.0	**1.0±0.0**	7.0±0.0	**2.8±1.0**	2.0±0.0	**3.3±0.9**	8.1±1.9	2.0±0.0	**0.7±0.5**	**2.8±1.0**	5.5±1.5	2.0±0.0	**3.3±0.9**	2.0±0.0	7.3±0.9
20%	Alg.1	**3.7±0.5**	2.0±0.0	**1.0±0.0**	6.7±0.5	**2.4±0.7**	2.0±0.0	**2.6±0.8**	7.0±1.5	2.0±0.0	**0.4±0.5**	**2.4±0.7**	4.8±1.2	2.0±0.0	**2.6±0.8**	2.0±0.0	6.6±0.8
	Alg.2	2.0±0.0	2.0±0.0	**1.0±0.0**	5.0±0.0	1.3±0.5	1.3±0.0	1.4±0.5	4.0±1.0	1.4±0.0	**0.4±0.5**	1.3±0.5	3.1±1.0	1.3±0.0	1.4±0.5	1.4±0.0	4.1±0.5
	Alg.3	**3.7±0.5**	**3.7±0.5**	**1.0±0.0**	**8.4±1.0**	**2.4±0.7**	**2.4±0.7**	**2.6±0.8**	**7.4±2.2**	**2.6±0.8**	**0.4±0.5**	**2.4±0.7**	**5.4±2.0**	**2.4±0.7**	**2.6±0.8**	**2.6±0.8**	**7.6±2.3**
	2WL	**3.7±0.5**	2.0±0.0	**1.0±0.0**	6.7±0.5	**2.4±0.7**	2.0±0.0	**2.6±0.8**	7.0±1.5	2.0±0.0	**0.4±0.5**	**2.4±0.7**	4.8±1.2	2.0±0.0	**2.6±0.8**	2.0±0.0	6.6±0.8
30%	Alg.1	**3.7±0.5**	2.0±0.0	**1.0±0.0**	6.7±0.5	**2.2±0.6**	2.0±0.0	**2.8±1.0**	7.0±1.6	2.0±0.0	**0.4±0.5**	**2.2±0.6**	4.6±1.1	2.0±0.0	**2.8±1.0**	2.0±0.0	6.8±1.0
	Alg.2	2.0±0.0	2.0±0.0	**1.0±0.0**	5.0±0.0	1.1±0.3	1.1±0.0	1.4±0.5	3.6±0.8	1.4±0.0	**0.4±0.5**	1.1±0.3	2.9±0.8	1.1±0.0	1.4±0.5	1.4±0.0	3.9±0.5
	Alg.3	**3.7±0.5**	**3.7±0.5**	**1.0±0.0**	**8.4±1.0**	**2.2±0.6**	**2.2±0.6**	**2.8±1.0**	**7.2±2.2**	**2.8±1.0**	**0.4±0.5**	**2.2±0.6**	**5.4±2.1**	**2.2±0.6**	**2.8±1.0**	**2.8±1.0**	**7.8±2.6**
	2WL	**3.7±0.5**	2.0±0.0	**1.0±0.0**	6.7±0.5	**2.2±0.6**	2.0±0.0	**2.8±1.0**	7.0±1.6	2.0±0.0	**0.4±0.5**	**2.2±0.6**	4.6±1.1	2.0±0.0	**2.8±1.0**	2.0±0.0	6.8±1.0

program on a personal computer and its specific configuration is OS: Microsoft WIN10; Processor: Intel[1] Core[2] i7-10750H CPU @ 2.60GHz; Memory: 32GB; Programming language: Python.

A. Experimental Design

To extensively validate the performance of different approaches, we randomly selected 12 datasets with different scales from UCI Repository (see https://www.uci.edu/) and carried out numerical experiments under different information granule clues, "/" means applying no work for the data, and "Discretization" is to discretize a range of numeric attributes into nominal attributes by the method shown in [23]. For convenience, the 12 converted datasets by nominal scale [17] are denoted datasets 1–12. The detailed information about the dataset is shown in Table I. To reduce the randomness of the experiment, we ran ten times on each dataset to obtain the average results.

B. Evaluating the Performance of Concept Learning

To experiment with the outcome of the approaches and make comparisons, we randomly take 2%, 5%, 10%, and

[1]Registered trademark.

[2]Trademarked.

TABLE IX

NUMBER OF TWO-WAY GRANULE CONCEPT (AVE.±SD) IN VOWEL

Objects		2%				5%				10%				15%			
Attribute		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$
15%	Alg.1	4.4±1.6	2.0±0.0	1.4±0.7	7.8±2.3	3.3±0.5	2.0±0.0	1.0±0.0	6.3±0.5	3.2±0.4	2.0±0.0	1.0±0.0	6.2±0.4	2.9±0.3	2.0±0.0	0.9±0.3	5.8±0.6
	Alg.2	3.4±1.6	2.9±0.0	0.9±0.3	7.2±1.9	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	1.9±0.3	1.9±0.0	0.9±0.3	4.7±0.6
	Alg.3	**6.0±2.0**	**4.2±1.0**	**2.3±1.1**	**12.5±4.1**	**4.4±1.0**	**3.6±0.5**	**1.5±0.5**	**9.5±2.0**	**3.6±0.8**	**3.2±0.4**	**1.2±0.4**	**8.0±1.6**	**3.5±1.0**	**3.2±0.6**	**1.2±0.6**	**7.9±2.2**
	2WL	4.4±1.6	2.0±0.0	**2.3±1.1**	8.7±2.7	3.3±0.5	2.0±0.0	**1.5±0.5**	6.8±1.0	3.2±0.4	2.0±0.0	**1.2±0.4**	6.4±0.8	2.9±0.3	2.0±0.0	**1.2±0.6**	6.1±0.9
20%	Alg.1	4.0±1.3	2.0±0.0	1.1±0.7	7.1±2.0	2.7±0.8	2.0±0.0	0.5±0.5	5.2±1.3	2.8±0.9	2.0±0.0	**0.5±0.5**	5.3±1.4	**2.5±0.8**	2.0±0.0	**0.3±0.5**	4.8±1.3
	Alg.2	2.5±1.3	2.3±0.0	0.9±0.5	5.7±1.8	1.5±0.5	1.5±0.0	0.5±0.5	3.5±1.0	1.5±0.5	1.5±0.0	**0.5±0.5**	3.5±1.0	1.3±0.5	1.3±0.0	**0.3±0.5**	2.9±1.0
	Alg.3	**4.7±2.0**	**4.2±1.4**	**1.3±1.0**	**10.2±4.4**	**3.2±1.3**	**2.9±0.9**	**0.6±0.7**	**6.7±2.9**	**2.9±1.0**	**2.6±0.7**	**0.5±0.5**	**6.0±2.2**	**2.5±0.8**	**2.3±0.5**	**0.3±0.5**	**5.1±1.8**
	2WL	4.0±1.3	2.0±0.0	**1.3±1.0**	7.3±2.3	2.7±0.8	2.0±0.0	0.6±0.7	5.3±1.5	2.8±0.9	2.0±0.0	**0.5±0.5**	5.3±1.4	**2.5±0.8**	2.0±0.0	**0.3±0.5**	4.8±1.3
30%	Alg.1	**3.1±0.9**	2.0±0.0	**0.6±0.5**	5.7±1.4	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.1±0.3**	2.0±0.0	**0.1±0.3**	4.2±0.6	**2.1±0.3**	2.0±0.0	**0.1±0.3**	4.2±0.6
	Alg.2	1.6±0.5	1.6±0.0	**0.6±0.5**	3.8±1.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0	1.1±0.3	1.1±0.0	**0.1±0.3**	2.3±0.6	1.1±0.3	1.1±0.0	**0.1±0.3**	2.3±0.6
	Alg.3	**3.1±0.9**	**3.1±0.9**	**0.6±0.5**	**6.8±2.3**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.1±0.3**	**2.1±0.3**	**0.1±0.3**	**4.3±0.9**	**2.1±0.3**	**2.1±0.3**	**0.1±0.3**	**4.3±0.9**
	2WL	**3.1±0.9**	2.0±0.0	**0.6±0.5**	5.7±1.4	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.1±0.3**	2.0±0.0	**0.1±0.3**	4.2±0.6	**2.1±0.3**	2.0±0.0	**0.1±0.3**	4.2±0.6

TABLE X

NUMBER OF TWO-WAY GRANULE CONCEPT (AVE.±SD) IN BANKNOTE

Objects		2%				5%				10%				15%			
Attribute		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$
15%	Alg.1	4.4±1.4	2.0±0.0	1.4±0.5	7.8±1.9	3.2±0.4	2.0±0.0	1.0±0.0	6.2±0.4	3.3±0.5	2.0±0.0	1.0±0.0	6.3±0.5	3.1±0.5	2.0±0.0	0.9±0.3	6.0±0.8
	Alg.2	3.2±1.5	2.8±0.0	0.9±0.3	6.9±1.8	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	1.9±0.3	1.9±0.0	0.9±0.3	4.7±0.6
	Alg.3	**5.6±1.9**	**4.1±0.7**	**1.9±1.0**	**11.6±3.6**	**4.4±0.9**	**3.7±0.5**	**1.5±0.5**	**9.6±1.9**	**3.7±0.8**	**3.2±0.4**	**1.2±0.4**	**8.1±1.6**	**3.3±0.8**	**3.0±0.4**	**1.0±0.4**	**7.3±1.6**
	2WL	4.4±1.4	2.0±0.0	**1.9±1.0**	8.3±2.4	3.2±0.4	2.0±0.0	**1.5±0.5**	6.7±0.9	3.3±0.5	2.0±0.0	**1.2±0.4**	6.5±0.9	3.1±0.5	2.0±0.0	**1.0±0.4**	6.1±0.9
20%	Alg.1	4.1±1.3	2.0±0.0	1.2±0.7	7.3±2.0	2.8±0.9	2.0±0.0	0.5±0.5	5.3±1.4	2.8±0.9	2.0±0.0	0.5±0.5	5.3±1.4	2.6±0.8	2.0±0.0	0.4±0.5	5.0±1.3
	Alg.2	2.6±1.4	2.4±0.0	0.9±0.7	5.9±2.1	1.5±0.5	1.5±0.0	0.5±0.5	3.5±1.0	1.5±0.5	1.5±0.0	0.5±0.5	3.5±1.0	1.4±0.5	1.4±0.0	0.4±0.5	3.2±1.0
	Alg.3	**4.8±1.8**	**4.1pm1.2**	**1.3±0.9**	**10.2±3.9**	**3.5±1.7**	**2.9±0.9**	**0.8±0.9**	**7.2±3.5**	**3.3±1.5**	**2.8±0.9**	**0.7±0.8**	**6.8±3.2**	**2.6±0.8**	**2.4±0.5**	**0.4±0.5**	**5.4±1.8**
	2WL	4.1±1.3	**4.1±1.3**	**1.3±0.9**	9.5±3.5	2.8±0.9	2.0±0.0	**0.8±0.9**	5.6±1.8	2.8±0.9	2.0±0.0	**0.7±0.8**	5.5±1.7	2.6±0.8	2.0±0.0	**0.4±0.5**	5.0±1.3
30%	Alg.1	1.3±0.9	2.0±0.0	0.6±0.5	3.9±1.4	2.0±0.0	2.0±0.0	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
	Alg.2	1.6±0.5	1.6±0.0	0.6±0.5	3.8±1.5	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0
	Alg.3	**3.1±0.9**	**3.1±0.9**	**0.6±0.5**	**6.8±2.3**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
	2WL	**3.1±0.9**	2.0±0.0	**0.6±0.5**	5.7±1.4	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**

TABLE XI

NUMBER OF TWO-WAY GRANULE CONCEPT (AVE.±SD) IN MUSHROOM

Objects		2%				5%				10%				15%			
Attribute		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$
15%	Alg.1	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0	**3.9±0.3**	2.0±0.0	1.0±0.0	6.9±0.3	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0
	Alg.2	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0
	Alg.3	**4.0±0.0**	**4.0±0.0**	**1.3±0.5**	**9.3±0.5**	**3.9±0.3**	**3.9±0.3**	**1.3±0.5**	**9.1±1.1**	**4.0±0.0**	**4.0±0.0**	**1.3±0.5**	**9.3±0.5**	**4.0±0.0**	**4.0±0.0**	**1.1±0.3**	**9.1±0.3**
	2WL	**4.0±0.0**	2.0±0.0	**1.3±0.5**	7.3±0.5	**3.9±0.3**	2.0±0.0	**1.3±0.5**	7.2±0.8	**4.0±0.0**	2.0±0.0	**1.3±0.5**	7.3±0.5	**4.0±0.0**	2.0±0.0	**1.1±0.3**	7.1±0.3
20%	Alg.1	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0
	Alg.2	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0
	Alg.3	**4.0±0.0**	**4.0±0.0**	**1.5±0.5**	**9.5±0.5**	**4.0±0.0**	**4.0±0.0**	**1.3±0.5**	**9.3±0.5**	**4.0±0.0**	**4.0±0.0**	**1.1±0.3**	**9.1±0.3**	**4.0±0.0**	**4.0±0.0**	**1.2±0.4**	**9.2±0.4**
	2WL	**4.0±0.0**	2.0±0.0	**1.5±0.5**	7.5±0.5	**4.0±0.0**	2.0±0.0	**1.3±0.5**	7.3±0.5	**4.0±0.0**	2.0±0.0	**1.1±0.3**	7.1±0.3	**4.0±0.0**	**4.0±0.0**	**1.2±0.4**	**9.2±0.4**
30%	Alg.1	**3.8±0.4**	2.0±0.0	1.0±0.0	6.8±0.4	**3.9±0.3**	2.0±0.0	1.0±0.0	6.9±0.3	**4.0±0.0**	2.0±0.0	1.0±0.0	7.0±0.0	**3.8±0.4**	2.0±0.0	1.0±0.0	6.8±0.4
	Alg.2	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0	2.0±0.0	2.0±0.0	1.0±0.0	5.0±0.0
	Alg.3	**3.8±0.4**	**3.8±0.4**	**1.5±0.5**	**9.1±1.3**	**3.9±0.3**	**3.9±0.3**	**1.4±0.5**	**9.2±1.1**	**4.0±0.0**	**4.0±0.0**	**1.4±0.5**	**9.4±0.5**	**3.8±0.4**	**3.8±0.4**	**1.4±0.5**	**9.0±1.3**
	2WL	**3.8±0.4**	2.0±0.0	**1.5±0.5**	7.3±0.9	**3.9±0.3**	2.0±0.0	**1.4±0.5**	7.3±0.8	**4.0±0.0**	2.0±0.0	**1.4±0.5**	6.4±0.5	**3.8±0.4**	2.0±0.0	**1.4±0.5**	7.2±0.9

15% objects in each dataset as the initial X_0 and 15%, 20%, and 30% attributes as the initial B_0. Specifically, we compare TCCL (i.e., Algorithms 1 and 2) and TCCL with concept movement (i.e., Algorithm 3) with the 2WL system [26]. Thus, the number of initial information granules is $4 * 3$ pairs in each dataset. In this section, we adopt the number of two-way granule concepts (including sufficient and necessary granule concept, sufficient granule concept, and necessary granule concept) and running time to evaluate the performance of concept learning.

1) Number of Two-Way Granule Concept: This part mainly verifies the number of two-way granule concepts generated by the three algorithms in our article. Next, the number of resulting two-way granule concepts of the datasets are shown in Tables II–XIII. The average (Ave.) value and standard deviation (SD) of the number of two-way granule concepts are shown in the table, where $\mathcal{G}$ is the sum of the number of granule concepts, that is, $|\mathcal{G}| = |\mathcal{G}_1| + |\mathcal{G}_2| + |\mathcal{G}_3|$. As seen from the resulting number of two-way granule concepts in Tables II–XIII, when given a dataset, for an arbitrary information granule, we can obtain the following conclusion.

1) The number of sufficient and necessary granule concepts is no more than 6 in Algorithms 1–3, and the 2WL method. This also verifies our conclusion in Propositions 1 and 2. Meanwhile, Algorithm 1 in our article has the same necessary, sufficient granule concept learning ability as the 2WL method in [26], but the whole granule concept learning ability of Algorithm 3 is better than that of it. Hence, we can say that TCCL with concept movement can perform better than 2WL and TCCL in two-way granule concept learning.
2) The sample size of initial information affects the number of two-way granule concepts, and as the size increases, the number of two-way granule concepts decreases. From the perspective of cognition, this is also consistent with the process of human cognition because the more initial cues, the more accurate the results.
3) In some cases, the failure to learn sufficient and necessary granule concepts is because the sample size of the initial clue is too large, and the information in the clue conflicts with each other, leading to the failure to learn sufficient and necessary granule concepts.

Meanwhile, we used Wilcoxon paired test to examine whether there exists a significant difference between Algorithm 3 and 2WL. Table XIV records the average number of two-way granule concepts and P-values. Given the test

TABLE XII

NUMBER OF TWO-WAY GRANULE CONCEPT (AVE.±SD) IN HTRU-2

Objects		2%				5%				10%				15%			
Attribute		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$
15%	Alg.1	**2.7±0.9**	2.0±0.0	**0.4±0.5**	5.1±1.4	**2.4±0.8**	2.0±0.0	**0.2±0.4**	4.6±1.2	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
	Alg.2	1.4±0.5	1.4±0.0	**0.4±0.5**	3.2±1.0	1.2±0.4	1.2±0.0	**0.2±0.4**	2.6±0.8	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0
	Alg.3	**2.7±0.9**	**2.7±0.9**	**0.4±0.5**	**5.8±2.3**	**2.4±0.8**	**2.4±0.8**	**0.2±0.4**	**5.0±2.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
	2WL	**2.7±0.9**	2.0±0.0	**0.4±0.5**	5.1±1.4	**2.4±0.8**	2.0±0.0	**0.2±0.4**	4.6±1.2	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
20%	Alg.1	**3.0±0.9**	2.0±0.0	**0.6±0.5**	5.6±1.4	**2.4±0.8**	2.0±0.0	**0.2±0.4**	4.6±1.2	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
	Alg.2	1.6±0.5	1.6±0.0	**0.6±0.5**	3.8±1.0	1.2±0.4	1.2±0.0	**0.2±0.4**	2.6±0.8	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0
	Alg.3	**3.0±0.9**	**3.0±0.9**	**0.6±0.5**	**6.6±2.3**	**2.4±0.8**	**2.4±0.8**	**0.2±0.4**	**5.0±2.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
	2WL	**3.0±0.9**	2.0±0.0	**0.6±0.5**	5.6±1.4	**2.4±0.8**	2.0±0.0	**0.2±0.4**	4.6±1.2	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
30%	Alg.1	**2.7±0.9**	2.0±0.0	**0.4±0.5**	5.1±1.4	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
	Alg.2	1.4±0.5	1.4±0.0	**0.4±0.5**	3.2±1.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0
	Alg.3	**2.7±0.9**	**2.7±0.9**	**0.4±0.5**	**5.8±2.3**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
	2WL	**2.7±0.9**	2.0±0.0	**0.4±0.5**	5.1±1.4	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**

TABLE XIII

NUMBER OF TWO-WAY GRANULE CONCEPT (AVE.±SD) IN SKIN SEGMENTATION

Objects		2%				5%				10%				15%			
Attribute		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}$
15%	Alg.1	2.4±0.5	2.0±0.0	0.4±0.5	4.8±1.0	2.2±0.4	2.0±0.0	0.2±0.4	4.4±0.8	2.1±0.3	2.0±0.0	**0.1±0.3**	4.2±0.6	2.2±0.4	2.0±0.0	**0.2±0.4**	4.4±0.8
	Alg.2	1.4±0.5	1.4±0.0	0.4±0.5	3.2±1.0	1.2±0.4	1.2±0.0	0.2±0.4	2.6±0.8	1.1±0.3	1.1±0.0	**0.1±0.3**	2.3±0.6	1.2±0.4	1.2±0.0	**0.2±0.4**	2.6±0.8
	Alg.3	**2.8±1.1**	**2.7±0.9**	**0.5±0.7**	**6.0±2.7**	**2.4±0.9**	**2.3±0.6**	**0.3±0.6**	**5.0±2.1**	**2.2±0.6**	**2.2±0.6**	**0.1±0.3**	**4.5±1.5**	**2.3±0.6**	**2.3±0.6**	**0.2±0.4**	**4.8±1.6**
	2WL	2.4±0.5	2.0±0.0	**0.5±0.7**	4.9±1.2	2.2±0.4	2.0±0.0	**0.3±0.6**	4.5±1.0	2.1±0.3	2.0±0.0	**0.1±0.3**	4.2±0.6	2.2±0.4	2.0±0.0	**0.2±0.4**	4.4±0.8
20%	Alg.1	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	4.0±0.0	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	2.0±0.0	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
	Alg.2	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0
	Alg.3	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
	2WL	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	4.0±0.0	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
30%	Alg.1	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	4.0±0.0	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
	Alg.2	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0	1.0±0.0	1.0±0.0	**0.0±0.0**	2.0±0.0
	Alg.3	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**
	2WL	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	4.0±0.0	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**	**2.0±0.0**	**2.0±0.0**	**0.0±0.0**	**4.0±0.0**

TABLE XIV

AVERAGE NUMBER OF TWO-WAY GRANULE CONCEPT AND WILCOXON PAIRED TEST RESULTS

Object Attribute	Method	2%	P-value	5%	P-value	10%	P-value	15%	P-value
15%	Alg.3	10.16	0.001	8.92	0.001	7.93	0.001	7.85	0.002
	2WL	7.53		6.85		6.22		6.25	
20%	Alg.3	9.86	0.001	8.11	0.001	7.48	0.004	7.26	0.008
	2WL	7.54		6.37		5.95		5.99	
30%	Alg.3	9.17	0.001	7.13	0.016	6.68	0.008	6.78	0.008
	2WL	7.03		5.97		5.49		5.77	

threshold is 0.05, we find that all the P-values are smaller than 0.05, which illustrates the significance between Algorithm 3 and 2WL.

2) Running Time: This part mainly verifies the running time of two-way granule concepts generated by the three algorithms in our article. Meanwhile, to reduce the randomness of the experiment, we still ran ten times on each dataset to obtain the average results. As seen from the comparison of running time for four methods (i.e., Algorithms 1–3, 2WL) in Fig. 5, when given a dataset, for an arbitrary information granule, we can obtain the following conclusion.

1) The running time of the four methods on the same datasets is related to the initial sample size. The larger the initial clue size is, the more time is needed.
2) The method proposed in this article saves more time than the 2WL method in [26] on the premise of ensuring learning ability. In particular, Algorithms 1 and 2 can be considered in particular decision-making because they can save much time.

C. Evaluating the Performance of Concept Evolution

Note that TCCL with concept movement can learn new granule concepts from one concept that has been learned (i.e., Algorithm 3). Next, we will verify the performance of concept movement in this section. Note that more time will be consumed when we verify the evolution ability of this article based on the existing concept learning method in [31] and [22], that is, we can continue to evolve the concept space via a concept movement viewpoint. Therefore, this article considers the number of granule concepts and the concept evolution ability of granule concepts to verify the effectiveness of concept evolution, in which the concept evolution ability refers to the ratio of the number of concepts compared. In this section, we first process the data according to the method of [31] and then randomly take 10%, 1%, and 0.1% objects as original information clues in datasets 1–5, datasets 6–9, and datasets 10–12, respectively.

TABLE XV

COMPARISON OF GRANULE CONCEPT NUMBER (AVE.±SD) IN 12 DATASETS

Method	CCLM	TWC	Alg.4		
No.s			$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$
Dataset 1	20.5±2.5	19.5±3.0	43.3±8.3	44.2±7.9	35.0±7.0
Dataset 2	61.1±2.0	61.3±2.1	217.7±20.0	206.8±21.4	205.7±29.5
Dataset 3	61.0±0.0	60.0±0.0	107.7±3.6	94.3±5.9	72.2±1.4
Dataset 4	28.7±2.6	24.8±3.0	47.8±9.5	45.5±8.9	41.3±8.9
Dataset 5	71.5±5.0	57.2±3.7	205.2±20.7	193.2±17.6	199.2±26.6
Dataset 6	32.4±1.2	33.7±2.0	68.5±6.2	68.1±6.5	58.6±7.6
Dataset 7	19.9±1.6	20.9±3.2	32.7±2.7	36.1±3.0	28.5±3.6
Dataset 8	54.2±2.7	33.7±1.9	91.6±6.1	86.0±6.1	82.8±6.2
Dataset 9	35.2±2.0	31.6±2.6	217.7±20.0	206.8±21.4	205.7±29.5
Dataset 10	47.3±5.2	30.3±5.0	109.9±16.2	98.4±13.3	100.6±17.7
Dataset 11	35.5±3.2	29.4±2.7	76.1±14.1	88.8±10.1	70.6±11.7
Dataset 12	40.8±2.9	31.2±4.3	63.5±7.3	54.3±6.6	50.9±5.4

This part mainly verifies the concept evolution ability of the proposed M-TCCL method (i.e., Algorithm 4) by analyzing the number of granule concepts and the evolution ability of granule concepts. Table XV records the average values of granule concepts and corresponding standard errors for three methods. From this table, we found that the number of granule concepts is more significant than CCLM and TWC methods on 12 datasets, which shows that M-TCCL can learn more granule concepts, that is, it has a more vital evolutionary ability compared with CCLM and TWC methods.

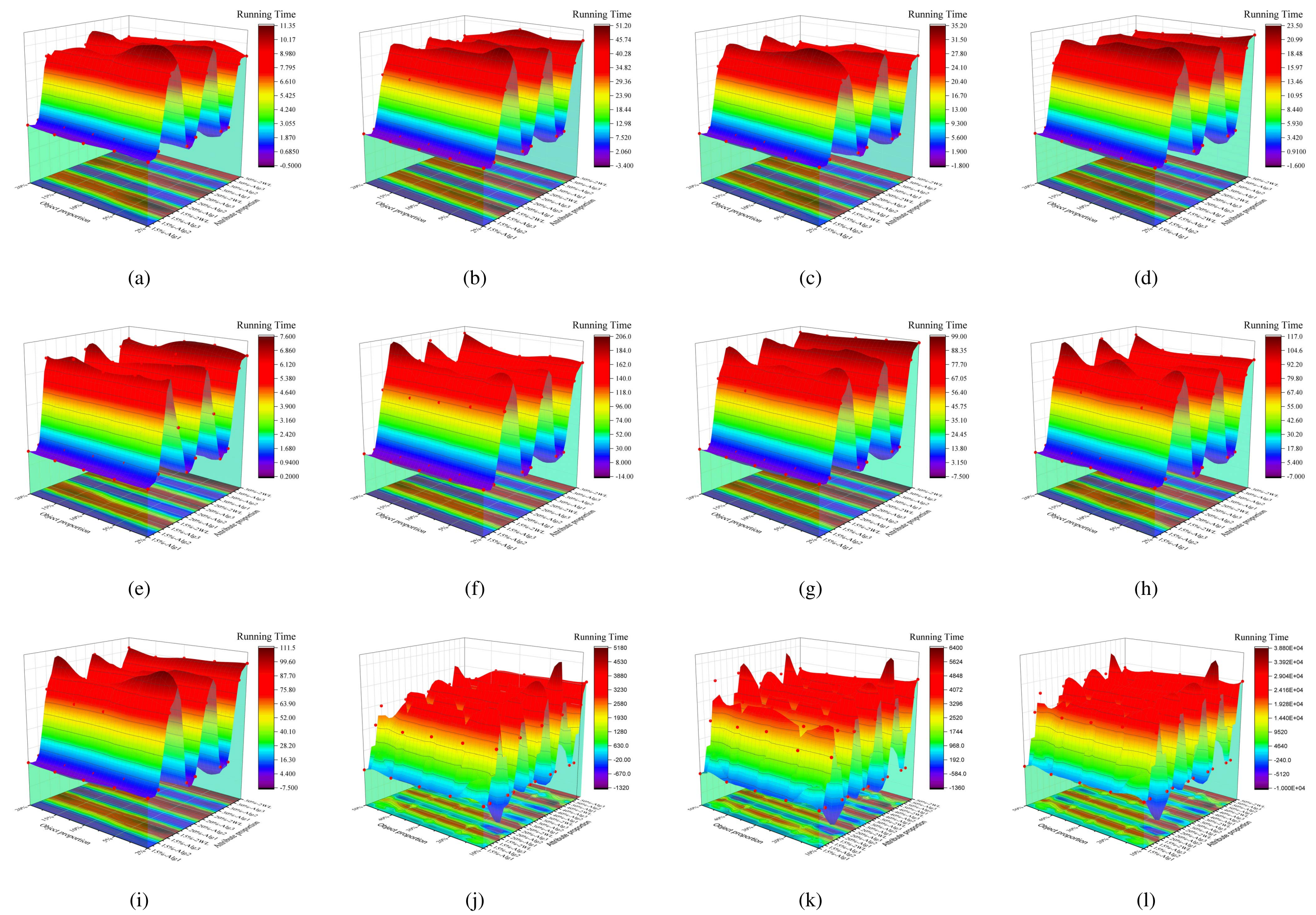

Fig. 5. Comparison of running time on 12 selected datasets. (a) Zoo. (b) Breast cancer. (c) Monks-2. (d) Glass identification. (e) Appendicitis. (f) Mammographic mass. (g) Tic-Tac-Toe. (h) Vowel. (i) Banknote. (j) Mushroom. (k) HTRU-2. (l) Skin segmentation.

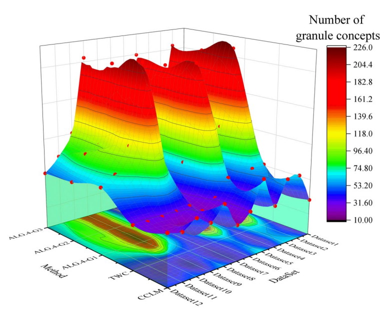

Fig. 6. Comparison of the number of granule concepts.

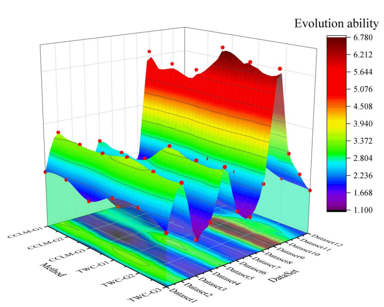

Fig. 7. Comparison of evolution ability of granule concepts.

Figs. 6 and 7 further show the advantages of M-TCCL in evolutionary ability. To test whether there is a significant difference between Algorithm 4 and the other two comparison algorithms, we take the Wilcoxon pairwise test and record the results in Table XVI. Given that the test threshold is 0.05, all the test P-values are 0.488×10^{-3}<0.05, and we could reject the null hypothesis (there is no difference between the two algorithms) and consider there is a significant difference between M-TCCL, CCLM, and TWC algorithms.

TABLE XVI

AVERAGE NUMBER OF GRANULE CONCEPT AND WILCOXON PAIRED TEST RESULTS BETWEEN THREE METHODS

Method	Alg.4			CCLM	TWC
	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$		
Ave.	101.98	97.91	93.41	40.78	32.89
P-value (Alg.4 vs CCLM)	0.488×10^{-3}	0.488×10^{-3}	0.488×10^{-3}	-	-
P-value (Alg.4 vs TWC)	0.488×10^{-3}	0.488×10^{-3}	0.488×10^{-3}	-	-

In addition, we evaluate the evolution ability of granule concepts and record the corresponding results in Table XVII. From this table, we know the granule concept evolution ability of Algorithm 4 to CCLM and TWC methods is all greater than 1. Meanwhile, we further adopt the T-test to examine whether the concept evolution ability is significantly greater than 1. All the test P-values recorded in Table XVII are smaller than 0.05, which means the concept evolution ability of Algorithm 4 is significantly more robust than the CCLM and TWC methods.

TABLE XVII
COMPARISON OF GRANULE CONCEPT EVOLUTION ABILITY IN 12 DATASETS

Method	ALg.4 / CCLM			ALg.4 / TWC		
No.s	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$
Dataset 1	2.11	2.16	1.71	2.22	2.27	1.79
Dataset 2	3.56	3.38	3.37	3.55	3.37	3.36
Dataset 3	1.77	1.55	1.18	1.80	1.57	1.20
Dataset 4	1.67	1.59	1.44	1.93	1.83	1.67
Dataset 5	2.87	2.70	2.79	3.59	3.38	3.48
Dataset 6	2.11	2.10	1.81	2.03	2.02	1.74
Dataset 7	1.64	1.81	1.43	1.56	1.73	1.36
Dataset 8	1.69	1.59	1.53	2.72	2.55	2.46
Dataset 9	6.18	5.88	5.84	6.89	6.54	6.51
Dataset 10	2.32	2.08	2.13	3.63	3.25	3.32
Dataset 11	2.14	2.50	1.99	2.59	3.02	2.40
Dataset 12	1.56	1.33	1.25	2.04	1.74	1.63
P-value	**0.003**	**0.003**	**0.009**	**0.001**	**0.001**	**0.004**

TABLE XVIII
COMPARATIVE ANALYSIS BETWEEN FIVE CCL MECHANISMS

	2WL	TWC	CCLM	TCCL	M-TCCL
Formal context	●	●	●	●	●
Learning of clues	●	○	○	●	●
Concept evolution	○	○	○	○	●
Granule concept space	○	○	●	●	●
Number of concept	M	S	S	M	L

Note: "●" denotes yes and "○" denotes no, "S" denotes the smaller number of granule concepts, "M" denotes the medium number of concepts, and "L" denotes the larger number of concepts.

D. Comparative Analysis Between Five CCL Mechanisms

According to the discussion in this section, we compare our method with other CCL mechanisms, including the 2WL mechanism [26], TWC mechanism [22], and CCLM mechanism [31]. Thus, we further make the comparative analysis between our mechanism (including TCCL and M-TCCL mechanisms) and others.

For clarity, all the similarities and differences are summed in Table XVIII. This table shows that the TCCL and M-TCCL mechanisms have the same characteristics except for the concept evolution. The five mechanisms consider the formal context, learning of clues (concept learning from given clues), concept evolution ability, granule concept space, and the number of concepts. These can also reflect that it is essential and reasonable to study our method to process the concept learning of given clues and evolute the concept space.

Both mechanisms utilize a pair of cognitive operators to learn the concept from the formal context. However, as shown in Table XVI, three main contributions and advantages are obtained in our model. First, our method introduces the 2WL method into concept-cognitive learning theory to learn the concept from a given clue (X, B), where the concept spaces of different cases (i.e., $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_1 \cap \mathcal{G}_2$) are also involved. Second, the TCCL can also learn the same concept as the 2WL, but it takes less time-consuming. Finally, the M-TCCL can continue to evolve concept space, which is more straightforward to show the effectiveness of TCCL with concept movement.

VII. CONCLUSION

TCCL is an effective cognitive system to describe the human cognitive process, and it can achieve the decision task by the given clues. The essence of TCCL is learning more from the unknown through a pair of cognitive operators and mapping it into granule concept space for different semantic interpretations (i.e., $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$). The movement viewpoint is mainly from the M-3WD model, the essential action through a pair of logical operators (i.e., $\vee$ and $\wedge$). The theory of TCCL has been rigorously verified, and simulations in real datasets also validate the practicability of TCCL in the current article.

Generally speaking, the sufficient and necessary granule concept is an exact cognitive concept from a given clue. In other words, it is crucial to learn pseudo-concept based on the known clue, especially in decision-making problems. The current article introduces a novel CCL system for concept cognitive and concept evolution. As an extension of 2WL, TCCL theory learning concepts from two ways, including top to bottom and bottom to top (i.e., a movement viewpoint). Furthermore, corresponding algorithms and experiments on various datasets demonstrate the effectiveness of the proposed TCCL compared with the 2WL system and CCLM methods.

The current TCCL just studies learning a granule concept from a given clue and the concept evolution based on the concept of what has been learned. So, there still exist some limitations that need to be concerned about, such as how to learn the concept from fuzzy data or interval data via our method, especially handling big data. Moreover, our method needs to consume more time in the process of concept evolution. Thus, how to improve the efficiency of concept evolution also deserves to be investigated. Our future work will continue to focus on these points.

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