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Dynamic updating variable precision three-way concept method based on two-way concept-cognitive learning in fuzzy formal contexts

Chengling Zhang [a], Eric C.C. Tsang [a,*], Weihua Xu [b], Yidong Lin [c], Lanzhen Yang [a], Jiaming Wu [a]

[a] School of Computer Science and Engineering, Macau University of Science and Technology, Taipa, Macau, China
[b] College of Artificial Intelligence, Southwest University, Chongqing, 400715, China
[c] School of Mathematics and Statistics, Minnan Normal University, Zhangzhou, 363000, China

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ABSTRACT

Cognitive learning and two-way learning are effective knowledge representations, which can simulate the human brain to learn concepts. The combination of both topics has achieved some results and improved conceptual evolution ability in fuzzy formal concept analysis (FFCA), but there are still some shortcomings: 1) FFCA does not consider the flexibility of concepts, which makes it difficult to select suitable concepts; 2) the existing necessary and sufficient concepts fail to learn directly from any information granules; 3) the cognitive learning mechanism ignores to integrate previously acquired knowledge into the present state in the process of dynamic concept learning. To tackle these issues, in this paper, we put forward a novel two-way concept-cognitive learning (TCCL) model based on three-way decision in fuzzy formal contexts. Firstly, we introduce the object and attribute operators to learn variable precision object induced three-way concept, where such concept has flexibility by adjusting thresholds. Then, to learn directly necessary and sufficient three-way concept from the given clues, we investigate a new TCCL model, which has low computation cost for concept learning. Furthermore, updating mechanism of three-way concept is discussed in dynamic learning environment. Finally, the conducted experiments explicate the effectiveness and feasibility of our proposed approach in the large-scale datasets.

1. Introduction

Cognitive computing is a computer system that simulates the human thinking process which includes perception, attention and thinking through computers [1]. It focuses on how to address the issues of inaccuracy, uncertainty and partially reality so as to achieve different levels of perception, memory and solution to the problems. Up to now, this theory has become a hot research direction that combines with other approaches such as psychology, mathematics, etc [2]. We have witnessed various theories and models of cognitive computing. For instance, concept-cognitive learning (CCL) reveals the systematic laws of human brain by concept formation and learning, such as Wille's concept [3,4], object-oriented concept [5], fuzzy concept [6–9] and three-way concept [10,11].

* Corresponding author.
E-mail addresses: chengling_zhang@126.com (C. Zhang), cctsang@must.edu.mo (E.C.C. Tsang), chxuwh@gmail.com (W. Xu), yidong_lin@yeah.net (Y. Lin), lzyang@hbu.edu.cn (L. Yang), wjmfuzzy@163.com (J. Wu).

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As an extension of Wille's concept (also called classical concept), the fuzzy concept learning can directly deal with the continuous data by transforming the attribute values into the membership degree of these attributes belonging to object, which greatly saves time consumption. After decades of development, fuzzy concept, as highly complementary to rough set, has attracted much attention including fuzzy two-way learning, concept clustering, rule extraction and so forth [12–21]. An outstanding advantage of fuzzy concept cannot only mine valuable information representation but also reduce the loss of useful information in the cognitive process. Moreover, three-way decision [23] is an innovative theory which facilitates thinking, problem settling and information processing in decision making [24]. Its basic idea of this theory is the tripartite granulation thinking paradigm, which is the positive region POS, the negative region NEG and the boundary region BND of a universal set. Combining three-way decision, three-way concept [25], as an extension of classical concept, can more comprehensively describe conceptual information by expressing the semantics of "jointly possessed" and "jointly not possessed" in a formal context. That is to say, three-way concept incorporates the positive region and negative region to divide objects (attributes) into three regions for making three-way decision. Moreover, several researches indicate that three-way decision is effective about conceptual representing and learning [10,26–28]. Nevertheless, these technologies have some limitations which manifest in: 1) although fuzzy concept can degenerate into classical concept, continuous data at this time will also be transformed into discrete data; 2) fuzzy concept shows great strictness when applying to crisp sets; 3) the classical concept emphasizes learning from discrete data, which is not suitable for continuous data without discretization.

CCL refers to adopt certain methods to learn unknown concepts from the given clues, which reveals human cognitive processes in the form of conceptual knowledge. In a general sense, it is generally investigated from three aspects: the cognitive mechanism of concept formation, the construction and optimization of concept cognition, and the simulation of cognitive process. Recently, many scholars have proposed several CCL models and methods to meet different practical requirements. For instance, Xu et al. [29] introduced a two-way learning model for transformation of information granules. Additionally, inspired by this work, to describe the transformation theory on fuzzy datasets, it was extended for fuzzy formal context [30]. Subsequently, Xu and Guo et al. [31,32] proposed a novel TCCL method for directly converting arbitrary information granules into sufficient and necessary information granules for dynamic concept learning, which improves the conceptual evolution mechanism. Shi and Mi et al. [33–35] constructed a series of CCL models, which aimed at obtaining conceptual generalization capability and solving classification task. These cognitive models have different cognitive mechanisms which include but not limited to conceptual clustering [18,34,35], dynamic concept learning [31,32,36–38] and classification learning [16,17,33]. Although these models achieve several significant advantages in two-way learning research, they still have some problems.

• The above existing CCL models emphasize how to acquire knowledge and their applications, but ignore how to select the suitable concepts from continuous data without discretization.

• Most existing CCL models do not think about the transformation of information granules based on three-way decision. Therefore, it is necessary for us to mine the wealth knowledge and further make decision analysis.

To address the aforementioned issues, introducing the idea of threshold is a significant tool for data mining and knowledge representation. We use medical decision-making as an example to introduce the main thought. Practically, when investigating suspected cases of the heart disease, doctors pay more attention to excluding whether new patients are diagnosed through the prime symptoms of previous patients, including shortness of breath, pectoralgia, nervous and so forth. If these patients' symptom indicators are higher than the standard values (thresholds), the doctor will primarily screening these indicators from new patient. In such case, a subset of patients who suffer from a particular disease are diagnosed. They are called the positive region. Simultaneously, if several subsidiary symptoms such as cough, abdominal distension and so on are not pivotal factors, when indexes are less than standard values, a subset of patients who are diagnosed not suffering from disease, is called the negative region. Furthermore, a subset of patients for whom the doctor is unable to make a clear diagnosis is called the boundary region. Therefore, a novel TCCL model is introduced through three-way decision. Fig. 1 describes the steps of the proposed approach in the block diagram. The main contributions of this paper are summarized as follows.

• By adjusting the thresholds α and γ, variable precision object induced three-way concept (for short VPO3C) has more flexibility in applications, where several suitable concepts are obtained by setting thresholds.

• To directly transform arbitrary information granules into necessary and sufficient three-way concepts, we propose a novel TCCL mechanism based on three-way decision. The experiment illustrates learning concept from the viewpoint of cognition in a shorter period of time.

• We propose an updating mechanism of VPO3C based on information granules and can integrate the newly input information into the current concepts to increase the efficiency of concept learning.

The remaining sections of this paper are structured as follows. Section 2 introduces some notions of **L**-context analysis, fuzzy formal concept analysis and fuzzy three-way concept analysis. Section 3 discusses the notions and properties of VPO3C. In Section 4, a novel TCCL based on VPO3C is presented. Section 5 studies an updating mechanism of VPO3C based on information granules in a dynamic environment. In Section 6, the experimental analysis is conducted to demonstrate the effectiveness of the proposed method. Section 7 covers some conclusions and further work.

2. Preliminaries

In this section, to begin with, we revisit the notion of fuzzy formal context and fuzzy logic, and more detailed information could be found from the corresponding papers [3,4,6,7].

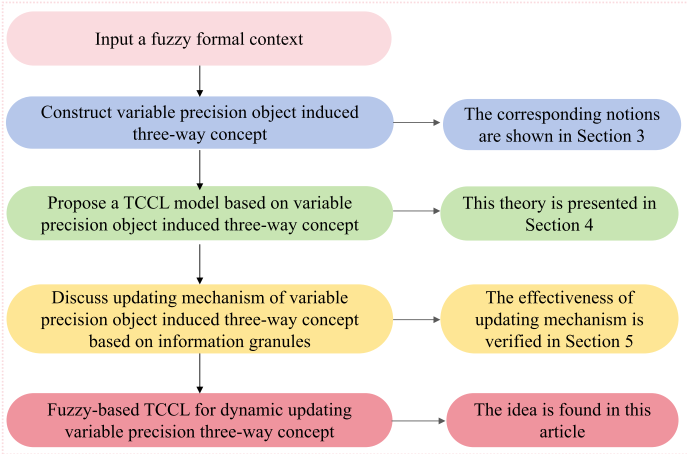

Fig. 1. Flowchart of the proposed approach.

2.1. **L**-context analysis

A structure $\mathbf{L} = (L, \vee, \wedge, \rightarrow, 0, 1)$ is called a residuated lattice such that 1) $(L, \vee, \wedge, 0, 1)$ is a lattice with the greatest element 1 and the least element 0; 2) $(L, \otimes, 1_L)$ is a commutative monoid (i.e., $\otimes$ is commutative and associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); 3) $(\otimes, \rightarrow)$ is an adjoint pair (i.e., $a \leq b \rightarrow c$ iff $a \otimes b \leq c$ holds for all $a, b, c \in L$).

In a residuated lattice **L** and a nonempty set G, a **L**-set of G is described as a mapping $\tilde{A} : G \rightarrow \mathbf{L}$ with $\tilde{A}(g)$ expressing as the truth degree of g belonging to $\tilde{A}$. Let a triple $(G, A, \tilde{R})$ be a **L**-context where $G = \{g_1, g_2, \cdots, g_n\}$ and $A = \{a_1, a_2, \cdots, a_m\}$ are the object set and attribute set, respectively. $\tilde{R} : G \times A \rightarrow L$ is a **L**-relation with $\tilde{R}(g, a)$, perceived as the truth value of object g having attribute a. In some practical applications, if $L = [0, 1]$, then **L**-context is denoted as a fuzzy formal context.

Definition 1. [4] Let $\psi : M \rightarrow N$ and $\varphi : N \rightarrow M$ be two mappings between ordered sets $(M, \leq)$ and $(N, \leq)$. Then a pair (ψ, φ) is called an isotone Galois connection satisfying

1. for any $m_1, m_2 \in M$, if $m_1 \leq m_2 \Rightarrow \psi(m_1) \leq \psi(m_2)$;
2. for any $n_1, n_2 \in N$, if $n_1 \leq n_2 \Rightarrow \varphi(n_1) \leq \varphi(n_2)$;
3. for any $m \in M$ and $n \in N$, $m \leq \varphi\big(\psi(m)\big)$ and $n \geq \psi\big(\varphi(n)\big)$.

2.2. Fuzzy formal concept analysis

Before starting with this subsection, we first briefly introduce the notations of fuzzy formal context and fuzzy concept.

Let U be a universe, and a fuzzy set $\tilde{F}$ on X which is described as a membership function $\tilde{F}(\cdot) : U \rightarrow [0, 1]$. For any $x \in U$, $\tilde{F}(x)$ is referred to as the membership degree of x with respect to $\tilde{F}$. In particular, we denote by $\mathcal{F}(U)$ the union of all fuzzy sets on U.

Let $(G, A, \tilde{R})$ be a fuzzy formal context, where G and A are the set of objects and the set of attributes, respectively. $\tilde{R}$ is a fuzzy relation between G and A, and each $\tilde{R}(g, a)$ represents the membership degree of object g to attribute a.

Definition 2. [22] Let $(G, A, \tilde{R})$ be a fuzzy formal context. For any $X \subseteq G$ and $\tilde{B} \in \mathcal{F}(A)$, two derivation operators $(\cdot)^*$ are denoted as follows:

$$X^*(a) = \bigwedge_{x \in X} \tilde{R}(x, a), a \in A,$$

$$\tilde{B}^* = \Big\{ x \in G : \forall a \in A, \tilde{B}(a) \leq \tilde{R}(x, a) \Big\},$$

where a pair $(X, \tilde{B})$ is called a fuzzy concept if $X^* = \tilde{B}$ and $\tilde{B}^* = X$. In a general sense, X and $\tilde{B}$ are the extent and intent of $(X, \tilde{B})$, respectively. It is evident that (X^{**}, X^*) and $(\tilde{B}^*, \tilde{B}^{**})$ are two fuzzy concepts.

2.3. Fuzzy three-way concept analysis

To maintain consistence, the above defined operators $*$ are called the positive operators in Definition 2. Correspondingly, reference [25] also gives a pair of negative operators and three-way operators as follows.

Let $\tilde{R}^c = G \times A - \tilde{R}$ be the complement of $\tilde{R}$, where $\tilde{R}^c(g, a)$ means that the non-membership degree of object g to attribute a.

Table 1
A fuzzy formal context.

G	a	b	c	d	e	f
1	0.35	1.00	0.10	0.35	0.90	0.50
2	0.80	0.45	1.00	0.20	0.50	0.65
3	0.25	0.80	0.50	0.25	0.85	0.80
4	0.35	0.60	0.20	0.30	0.35	0.65
5	0.30	0.75	0.85	0.20	0.80	0.70

Given a fuzzy formal context $(G, A, \tilde{R})$, for any $X \subseteq G$ and $\tilde{B} \in \mathcal{F}(A)$, a pair of operators $(\cdot)^{\bar{*}}$ are given as follows:

$$X^{\bar{*}}(a) = \bigwedge_{x\in X} \tilde{R}^c(x,a), a \in A,$$

$$\tilde{B}^{\bar{*}} = \left\{ x \in G : \forall a \in A, \tilde{B}(a) \le \tilde{R}^c(x,a) \right\},$$

this pair of negative operators could describe the information between objects and attributes. In addition, $X^{\bar{*}}$ represents that X possesses the minimum non-membership degree on attribute a. $\tilde{B}^{\bar{*}}$ is considered to be the cognitive procee from attribute to object. As described previously, both the positive operators and negative operators are equally important to characterize a concept. Thus, three-way operator based on three-way decision is constructed and the corresponding three-way concept is formed. For $X \subseteq G$ and $B, C \subseteq A$, the three-way operators $\lessdot: \mathcal{P}(G) \to \mathcal{F}(A) \times \mathcal{F}(A)$ and $\gtrdot: \mathcal{F}(A) \times \mathcal{F}(A) \to \mathcal{P}(G)$ are defined by $X^{\lessdot} = (X^*, X^{\bar{*}})$ and $(\tilde{B}, \tilde{C})^{\gtrdot} = \tilde{B}^* \cap \tilde{C}^{\bar{*}}$. Then $(X, (\tilde{B}, \tilde{C}))$ is a fuzzy three-way concept if $X^{\lessdot} = (\tilde{B}, \tilde{C})$ and $(\tilde{B}, \tilde{C})^{\gtrdot} = X$.

Example 1. Table 1 $(G, A, \tilde{R})$ is a record of screening patients for heart disease, where G and A are five patients and six symptoms. For the sake of simplicity, we denote five patients by 1, 2, 3, 4 and 5, respectively, and six symptoms (cough, shortness of breath, nervous, fever, edema and pectoralgia) by a, b, c, d, e and f, respectively. We take $X = \{2, 3\}$ and compute $X^* = (a^{0.25}, b^{0.45}, c^{0.5}, d^{0.2}, e^{0.5}, f^{0.65})$. After that $X^{**} = \{2, 3, 5\}$ and then (X^{**}, X^*) is a fuzzy concept. In addition, $X^{\bar{*}} = (a^{0.2}, b^{0.2}, c^0, d^{0.75}, e^{0.15}, f^{0.2})$ which implies that $X^{\bar{*}\bar{*}} = \{2, 3, 5\}$. Thus, $(\{2, 3, 5\}, ((a^{0.25}, b^{0.45} c^{0.5}, d^{0.2}, e^{0.5}, f^{0.65}), (a^{0.2}, b^{0.2} c^0, d^{0.75}, e^{0.15}, f^{0.2})))$ is a fuzzy three-way concept.

3. The construction process of VPO3C

In this section, we first propose two categories of concepts which are α positive concept and γ negative concept. Then several properties are discussed as follows. Subsequently, inspired by the thought of three-way decision [23], we introduce the notions of VPO3C.

3.1. α positive concept and γ negative concept

Definition 3. Let $(G, A, \tilde{R})$ be a fuzzy formal context. For any $X \subseteq G$, $B \subseteq A$, two operators $\uparrow_\alpha$ and $\downarrow_\alpha$ are denoted as:

$$X^{\uparrow_\alpha} = \{a \in A : \forall g \in X^c, \tilde{R}(g,a) \ge \alpha\};$$

$$B^{\downarrow_\alpha} = \{g \in G : \exists a \in B, \tilde{R}(g,a) < \alpha\},$$

where $X^{\uparrow_\alpha}$ is the set of attributes shared by all objects in the complement of X with the degree not less than α, and $B^{\downarrow_\alpha}$ is the set of objects possessing at least one attribute in B with the degree less than α. For convenience, we denote $\uparrow_\alpha$ and $\downarrow_\alpha$ as the α-positive operators.

Property 1. *Let $(G, A, \tilde{R})$ be a fuzzy formal context. $X, X_1, X_2, X_i \subseteq G$, $B, B_1, B_2, B_i \subseteq A$ ($i \in \Lambda$ where Λ is an index set). Then we have*

1. $X_1 \subseteq X_2 \Rightarrow X_1^{\uparrow_\alpha} \subseteq X_2^{\uparrow_\alpha}$;
2. $B_1 \subseteq B_2 \Rightarrow B_1^{\downarrow_\alpha} \subseteq B_2^{\downarrow_\alpha}$;
3. $X^{\uparrow_\alpha\downarrow_\alpha} \subseteq X$, $B \subseteq B^{\downarrow_\alpha\uparrow_\alpha}$;
4. $X^{\uparrow_\alpha\downarrow_\alpha\uparrow_\alpha} = X^{\uparrow_\alpha}$, $B^{\downarrow_\alpha} = B^{\downarrow_\alpha\uparrow_\alpha\downarrow_\alpha}$;
5. $X^{\uparrow_\alpha} \supseteq B \Leftrightarrow X \supseteq B^{\downarrow_\alpha}$;
6. $(\bigcap_{i\in\Lambda} X_i)^{\uparrow_\alpha} = \bigcap_{i\in\Lambda} X_i^{\uparrow_\alpha}$, $(\bigcup_{i\in\Lambda} B_i)^{\downarrow_\alpha} = \bigcup_{i\in\Lambda} B_i^{\downarrow_\alpha}$;
7. $(\bigcup_{i\in\Lambda} X_i)^{\uparrow_\alpha} \supseteq \bigcup_{i\in\Lambda} X_i^{\uparrow_\alpha}$, $(\bigcap_{i\in\Lambda} B_i)^{\downarrow_\alpha} \subseteq \bigcap_{i\in\Lambda} B_i^{\downarrow_\alpha}$.

Proof. 1. For any $X_1 \subseteq X_2$, it is evident that $X_1^{\uparrow_\alpha} \subseteq X_2^{\uparrow_\alpha}$ with the fact of $X_2^c \subseteq X_1^c$.
2. Easy, so it is omitted.
3. Proceeding by contradiction, suppose that $X^{\uparrow_\alpha\downarrow_\alpha} \subseteq X$ does not hold. There is $y \in X^{\uparrow_\alpha\downarrow_\alpha}$ such that $y \notin X$. Assume that $y \in X^{\uparrow_\alpha\downarrow_\alpha}$, then there exists $a \in X^{\uparrow_\alpha}$ satisfying $\tilde{R}(y,a) < \alpha$. With the fact of $a \in X^{\uparrow_\alpha}$, we have $\tilde{R}(x,a) \ge \alpha$ for any $x \in X^c$. As $y \notin X$, then $y \in X^c$. Hence, $\tilde{R}(y,a) \ge \alpha$. This is a contradiction with the fact of $\tilde{R}(y,a) < \alpha$. Thus, $X^{\uparrow_\alpha\downarrow_\alpha} \subseteq X$. We next verify the another formula. For

any $a \notin B^{\downarrow_\alpha \uparrow_\alpha}$, then there exists $x \notin B^{\downarrow_\alpha}$ such that $\tilde{R}(x,a) < \alpha$. With the fact of $x \notin B^{\downarrow_\alpha}$, we have $\tilde{R}(x,b) \geq \alpha$ for any $b \in B$. This implies that $a \notin B$. Therefore, $B \subseteq B^{\downarrow_\alpha \uparrow_\alpha}$.

4. For any $X \subseteq G$, since $X^{\uparrow_\alpha \downarrow_\alpha} \subseteq X$, it follows that $X^{\uparrow_\alpha \downarrow_\alpha \uparrow_\alpha} \subseteq X^{\uparrow_\alpha}$ from Property 1.1 and 1.3. Conversely, suppose that $B = X^{\uparrow_\alpha}$. Because $B \subseteq B^{\downarrow_\alpha \uparrow_\alpha}$ from Property 1.3, we have $X^{\uparrow_\alpha} \subseteq X^{\uparrow_\alpha \downarrow_\alpha \uparrow_\alpha}$, which follows that $X^{\uparrow_\alpha} = X^{\uparrow_\alpha \downarrow_\alpha \uparrow_\alpha}$. In a similar way, it can be proved that $B\downarrow_\alpha = B^{\downarrow_\alpha \uparrow_\alpha \downarrow_\alpha}$.
5. Necessity. Since $X^{\uparrow_\alpha} \supseteq B$, we have $X \supseteq X^{\uparrow_\alpha \downarrow_\alpha} \supseteq B^{\downarrow_\alpha}$ from Property 1.1 and 1.3. Sufficiency. Thanks to $X \supseteq B^{\downarrow_\alpha}$, it holds that $X^{\uparrow_\alpha} \supseteq B^{\downarrow_\alpha \uparrow_\alpha} \supseteq B$.
6. The following statements are equivalent:

$$\begin{aligned} a \in (\bigcap_{i\in\Lambda} X_i)^{\uparrow_\alpha} &\Leftrightarrow \forall g \in (\bigcap_{i\in\Lambda} X_i)^c, \tilde{R}(g,a) \geq \alpha \\ &\Leftrightarrow \forall g \in \bigcup_{i\in\Lambda} X_i^c, \tilde{R}(g,a) \geq \alpha \\ &\Leftrightarrow \forall i \in \Lambda, \forall g \in X_i^c, \tilde{R}(g,a) \geq \alpha \\ &\Leftrightarrow \forall i \in \Lambda, a \in X_i^{\uparrow_\alpha} \\ &\Leftrightarrow a \in \bigcap_{i\in\Lambda} X_i^{\uparrow_\alpha}. \end{aligned}$$

It follows that $(\bigcap_{i\in\Lambda} X_i)^{\uparrow_\alpha} = \bigcap_{i\in\Lambda} X_i^{\uparrow_\alpha}$. The other one is similarly proved.
7. It can be obviously induced from Property 1.1. □

From the above analysis, we see that the pair $(\uparrow_\alpha, \downarrow_\alpha)$ forms an isotone Galois connection between $(2^G, \subseteq)$ and $(2^A, \subseteq)$ from Definition 1 based on Property 1.1, 1.2 and 1.3. Then item 4 shows that the result of applying $\uparrow_\alpha$ and $\downarrow_\alpha$ three times respectively is the same as that of applying them once. Item 5 is an equivalent statement of isotone Galois connection with respect to $(\uparrow_\alpha, \downarrow_\alpha)$. Items 6 and 7 demonstrate that the distribution property derived from an object set is valid in the intersection but not in the union. Conversely, for any subset of attributes, the distribution property is applicable in the union but not in the intersection.

Definition 4. Let $(G, A, \tilde{R})$ be a fuzzy formal context. A pair (X, B) satisfying $X^{\uparrow_\alpha} = B$ and $B^{\downarrow_\alpha} = X$ is defined as a α-positive concept. Then X and B are the extent and intent of α-positive concept.

For any $X \subseteq G$, $B \subseteq A$, we see that $(X^{\uparrow_\alpha \downarrow_\alpha}, X^{\uparrow_\alpha})$ and $(B^{\downarrow_\alpha}, B^{\downarrow_\alpha \uparrow_\alpha})$ are α-positive concepts from Property 1.4. Subsequently, for any two α-positive concepts (X_1, B_1) and (X_2, B_2), the partial order $\leq$ is denoted as follows:

$$(X_1, B_1) \leq (X_2, B_2) \Leftrightarrow X_1 \subseteq X_2 \;\Leftrightarrow B_1 \subseteq B_2,$$

where (X_1, B_1) is a sub-concept of (X_2, B_2), and (X_2, B_2) is referred to as a super-concept of (X_1, B_1). Then the set of all α-positive concepts forms a complete lattice, denoted by $L_\alpha(G, A, \tilde{R})$.

Example 2. Further considering Table 1, given $\alpha = 0.6$, for the object $X = \{2, 3\}$, this implies that doctor mainly focuses on symptoms based on the medical reports of previous patients 1, 4 and 5 when examining patients 2 and 3. Then we can obtain $X^{\uparrow_\alpha} = \{b\}$, which means that the main symptom of heart disease caused by patients 1, 4 and 5 is shortness of breath b when screening for heart disease. Now, we focus on whether patients 2 and 3 have symptom b. On this basis, we compute $X^{\uparrow_\alpha \downarrow_\alpha} = \{2\}$. This shows that only patient 2 has non-prominent symptoms among the two patients 2 and 3. That is to say, patient 2 appears to have milder symptoms of heart disease compared to patient 3.

On the other hand, concept plays a crucial role in solving classification problems [16–18,33,34] in concept-cognitive learning. According to the property of α-positive concept, we know that $\frac{|X^{\uparrow_\alpha \downarrow_\alpha}|}{|X|} \leq 1$. However, it is well known that $\frac{|X^{**}|}{|X|} \geq 1$ from Definition 2. Therefore, the extent of α-positive concept is in the same class X, which is more accurate than fuzzy concept.

Henceforth, we will continue to introduce the notion of γ-negative concept.

Definition 5. Let $(G, A, \tilde{R})$ be a fuzzy formal context. For any $X \subseteq G$, $B \subseteq A$, two operators $\uparrow_\gamma$ and $\downarrow_\gamma$ are denoted as:

$$X^{\uparrow_\gamma} = \{a \in A : \forall g \in X^c, \tilde{R}(g,a) < \gamma\};$$

$$B^{\downarrow_\gamma} = \{g \in G : \exists a \in B, \tilde{R}(g,a) \geq \gamma\},$$

where $X^{\uparrow_\gamma}$ is the set of attributes possessed by all objects in the complement of X with the degree less than γ, and $B^{\downarrow_\gamma}$ is the set of objects possessing at least one attribute in B with the degree not less than γ. For the sake of simplicity, we write $\uparrow_\gamma$ and $\downarrow_\gamma$ as the γ-negative operators.

Analogously, we also discuss its properties, but its proof process is similar to Property 1, so we will omit them. For $X, X_1, X_2, X_i \subseteq G$, $B, B_1, B_2, B_i \subseteq A$ ($i \in \Lambda$ where Λ is an index set), we have

1. $X_1 \subseteq X_2 \Rightarrow X_1^{\uparrow_\gamma} \subseteq X_2^{\uparrow_\gamma}$;
2. $B_1 \subseteq B_2 \Rightarrow B_1^{\downarrow_\gamma} \subseteq B_2^{\downarrow_\gamma}$;
3. $X^{\uparrow_\gamma\downarrow_\gamma} \subseteq X,\ B \subseteq B^{\downarrow_\gamma\uparrow_\gamma}$;
4. $X^{\uparrow_\gamma\downarrow_\gamma\uparrow_\gamma} = X^{\uparrow_\gamma},\ B^{\downarrow_\gamma} = B^{\downarrow_\gamma\uparrow_\gamma\downarrow_\gamma}$;
5. $X^{\uparrow_\gamma} \supseteq B \Leftrightarrow X \supseteq B^{\downarrow_\gamma}$;
6. $(\bigcap_{i\in\Lambda} X_i)^{\uparrow_\gamma} = \bigcap_{i\in\Lambda} X_i^{\uparrow_\gamma},\ (\bigcup_{i\in\Lambda} B_i)^{\downarrow_\gamma} = \bigcup_{i\in\Lambda} B_i^{\downarrow_\gamma}$;
7. $(\bigcup_{i\in\Lambda} X_i)^{\uparrow_\gamma} \supseteq \bigcup_{i\in\Lambda} X_i^{\uparrow_\gamma},\ (\bigcap_{i\in\Lambda} B_i)^{\downarrow_\gamma} \subseteq \bigcap_{i\in\Lambda} B_i^{\downarrow_\gamma}$.

Moreover, if $X^{\uparrow_\gamma} = B$ and $B^{\downarrow_\gamma} = X$ for any $X \subseteq G,\ B \subseteq A$, then (X, B) is called a γ-negative concept.

Example 3. Further continuing Table 1, suppose that $X = \{2, 3\}$ and $\gamma = 0.4$, it can be checked that $X^{\uparrow_\gamma} = \{a, d\}$ which represents that patients 1, 4 and 5 exhibit milder symptoms cough a and fever d when screening for heart disease. To assess whether patients 2 and 3 have symptoms a and d. Then we can obtain $\{a, d\}^{\downarrow_\gamma} = \{2\}$. This implies that patient 2 appears to be more severe symptoms compared to patient 3. We use $(2, ad)$ instead of its standard format $(2, \{a, d\})$. In the end, $(2, ad)$ is a γ-concept.

3.2. Variable precision object induced three-way concept

To address the aforementioned challenges, three-way decision plays a pivotal role for data mining and decision-making activities. Inspired by the thought of three-way concept [10], we will propose the notion of variable precision object induced three-way concept in this subsection.

For two pairs of sets (B_i, C_i) and (B_j, C_j), if $B_i \subseteq B_j$ and $C_i \subseteq C_j$, then we denote by $(B_i, C_i) \subseteq (B_j, C_j)$. Furthermore, the union and intersection are given by [25]:

$$(B_i, C_i) \cup (B_j, C_j) = (B_i \cup B_j, C_i \cup C_j);$$
$$(B_i, C_i) \cap (B_j, C_j) = (B_i \cap B_j, C_i \cap C_j).$$

Definition 6. Let $(G, A, \tilde{R})$ be a fuzzy formal context with $0 \le \gamma < \alpha \le 1$. For any $X \subseteq G,\ B, C \subseteq A$, a pair of three-way operators is defined as follows:

$$\begin{aligned} &X^{\lessdot_{\alpha\gamma}} = (X^{\uparrow_\alpha}, X^{\uparrow_\gamma}); \\ &(B, C)^{\gtrdot_{\alpha\gamma}} = B^{\downarrow_\alpha} \cup C^{\downarrow_\gamma}, \end{aligned} \tag{1}$$

where this pair of three-way operators is called the $\alpha - \gamma$-object induced three-way operator or variable precision object induced three-way operator.

It should be noticed that the condition $0 \le \gamma < \alpha \le 1$ can guarantee the disjointness of $X^{\uparrow_\alpha}$ and $X^{\downarrow_\gamma}$. Moreover, the operators $\lessdot_{\alpha\gamma}$ and $\gtrdot_{\alpha\gamma}$ combine α-positive operators and γ-negative operators together, which describe not only the positive attributes but also the negative attributes. In such case, for an arbitrary $X \subseteq G$, $X^{\lessdot_{\alpha\gamma}}$ can formalize A into three disjoint regions, that is the positive, negative and boundary regions:

(1) $POS_\alpha = X^{\uparrow_\alpha} = \{a \in A : \forall g \in X^c, \tilde{R}(g, a) \ge \alpha\}$;

(2) $NEG_\gamma = X^{\uparrow_\gamma} = \{a \in A : \forall g \in X^c, \tilde{R}(g, a) < \gamma\}$;

(3) $BND_{\alpha,\gamma} = A - POS_\alpha - NEG_\gamma$.

This pair of $\alpha - \gamma$-object induced three-way operator has the following properties.

Property 2. *Let $(G, A, \tilde{R})$ be a fuzzy formal context with $0 \le \gamma < \alpha \le 1$. For $X, X_1, X_2, X_i \subseteq G,\ B, C, B_1, C_1, B_2, C_2, B_i, C_i \subseteq A$ ($i \in \Lambda$ where Λ is an index set). Then we have*

1. $X_1 \subseteq X_2 \Rightarrow X_1^{\lessdot_{\alpha\gamma}} \subseteq X_2^{\lessdot_{\alpha\gamma}}$;
2. $(B_1, C_1) \subseteq (B_2, C_2) \Rightarrow (B_1, C_1)^{\gtrdot_{\alpha\gamma}} \subseteq (B_2, C_2)^{\gtrdot_{\alpha\gamma}}$;
3. $X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}} \subseteq X,\ (B, C) \subseteq (B, C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}}$;
4. $X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}} = X^{\lessdot_{\alpha\gamma}},\ (B, C)^{\gtrdot_{\alpha\gamma}} = (B, C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}$;
5. $X^{\lessdot_{\alpha\gamma}} \supseteq (B, C) \Leftrightarrow X \supseteq (B, C)^{\gtrdot_{\alpha\gamma}}$;
6. $(\bigcap_{i\in\Lambda} X_i)^{\lessdot_{\alpha\gamma}} = \bigcap_{i\in\Lambda} X_i^{\lessdot_{\alpha\gamma}},\ (\bigcup_{i\in\Lambda}(B_i, C_i))^{\gtrdot_{\alpha\gamma}} = \bigcup_{i\in\Lambda}(B_i, C_i)^{\gtrdot_{\alpha\gamma}}$;
7. $(\bigcup_{i\in\Lambda} X_i)^{\lessdot_{\alpha\gamma}} \supseteq \bigcup_{i\in\Lambda} X_i^{\lessdot_{\alpha\gamma}},\ (\bigcap_{i\in\Lambda}(B_i, C_i))^{\gtrdot_{\alpha\gamma}} \subseteq \bigcap_{i\in\Lambda}(B_i, C_i)^{\gtrdot_{\alpha\gamma}}$.

Proof. We only prove the items 3, 4 and 6 here. The conclusions 1, 2, 5 and 7 are obvious.

3. From Property 1.3, we obtain $X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}} = (X^{\uparrow_\alpha}, X^{\uparrow_\gamma})^{\gtrdot_{\alpha\gamma}} = X^{\uparrow_\alpha\downarrow_\alpha} \cup X^{\uparrow_\gamma\downarrow_\gamma} \subseteq X \cup X = X$. Analogously, $(B, C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}} = (B^{\downarrow_\alpha} \cup C^{\downarrow_\gamma})^{\lessdot_{\alpha\gamma}} = ((B^{\downarrow_\alpha} \cup C^{\downarrow_\gamma})^{\uparrow_\alpha}, (B^{\downarrow_\alpha} \cup C^{\downarrow_\gamma})^{\uparrow_\gamma}) \supseteq (B^{\downarrow_\alpha\uparrow_\alpha} \cup C^{\downarrow_\gamma\uparrow_\alpha}, B^{\downarrow_\alpha\uparrow_\gamma} \cup C^{\downarrow_\gamma\uparrow_\gamma}) \supseteq (B^{\downarrow_\alpha\uparrow_\alpha}, C^{\downarrow_\gamma\uparrow_\gamma}) \supseteq (B, C)$.

4. It follows from items 1 and 3 that $X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}} \subseteq X^{\lessdot_{\alpha\gamma}}$. On the other hand,

$$X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}} = (X^{\uparrow_\alpha}, X^{\uparrow_\gamma})^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}} = (X^{\uparrow_\alpha\downarrow_\alpha} \cup X^{\uparrow_\gamma\downarrow_\gamma})^{\lessdot_{\alpha\gamma}}$$
$$= ((X^{\uparrow_\alpha\downarrow_\alpha} \cup X^{\uparrow_\gamma\downarrow_\gamma})^{\uparrow_\alpha}, (X^{\uparrow_\alpha\downarrow_\alpha} \cup X^{\uparrow_\gamma\downarrow_\gamma})^{\uparrow_\gamma})$$
$$\supseteq (X^{\uparrow_\alpha\downarrow_\alpha\uparrow_\alpha} \cup X^{\uparrow_\gamma\downarrow_\gamma\uparrow_\alpha}, X^{\uparrow_\alpha\downarrow_\alpha\uparrow_\gamma} \cup X^{\uparrow_\gamma\downarrow_\gamma\uparrow_\gamma}))$$
$$\supseteq (X^{\uparrow_\alpha\downarrow_\alpha\uparrow_\alpha}, X^{\uparrow_\gamma\downarrow_\gamma\uparrow_\gamma}) = (X^{\uparrow_\alpha}, X^{\uparrow_\gamma}) = X^{\lessdot_{\alpha\gamma}}.$$

Thus, we have $X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}} = X^{\lessdot_{\alpha\gamma}}$. By the similar proof, we prove $(B,C)^{\gtrdot_{\alpha\gamma}} = (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}$.

6. It holds that $(\bigcap_{i\in\Lambda} X_i)^{\lessdot_{\alpha\gamma}} = ((\bigcap_{i\in\Lambda} X_i)^{\uparrow_\alpha}, (\bigcap_{i\in\Lambda} X_i)^{\uparrow_\gamma}) = (\bigcap_{i\in\Lambda} X_i^{\uparrow_\alpha}, \bigcap_{i\in\Lambda} X_i^{\uparrow_\gamma}) = \bigcap_{i\in\Lambda}(X_i^{\uparrow_\alpha}, X_i^{\uparrow_\gamma}) = \bigcap_{i\in\Lambda} X_i^{\lessdot_{\alpha\gamma}}$ In a similar way, it is obvious that $(\bigcup_{i\in\Lambda}(B_i,C_i))^{\gtrdot_{\alpha\gamma}} = \bigcup_{i\in\Lambda}(B_i,C_i)^{\gtrdot_{\alpha\gamma}}$ also holds. □

Based on the $\alpha-\gamma$-object induced three-way operator, we can define the following three-way concept.

Definition 7. Let $(G,A,\tilde{R})$ be a fuzzy formal context with $0 \le \gamma < \alpha \le 1$. For any $X \subseteq G,\ B,C \subseteq A$, if a pair $(X,(B,C))$ satisfying $X^{\lessdot_{\alpha\gamma}} = (B,C)$ and $(B,C)^{\gtrdot_{\alpha\gamma}} = X$ is named a $\alpha-\gamma$-object induced three-way concept or variable precision object induced three-way concept (short for VPO3C). Furthermore, it is known that $(X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, X^{\lessdot_{\alpha\gamma}})$ and $((B,C)^{\gtrdot_{\alpha\gamma}}, (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}})$ are VPO3Cs from Property 2.4. Then the set of all VPO3Cs is denoted by $L_{\alpha\gamma}(G,A,\tilde{R})$, where the infimum and supermum are respectively given by:

$$\bigwedge_{i\in\Lambda}\Big(X_i,(B_i,C_i)\Big) = \Big((\bigwedge_{i\in\Lambda} X_i)^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, \bigwedge_{i\in\Lambda}(B_i,C_i)\Big);$$
$$\bigvee_{i\in\Lambda}\Big(X_i,(B_i,C_i)\Big) = \Big(\bigvee_{i\in\Lambda} X_i, (\bigvee_{i\in\Lambda}(B_i,C_i))^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}}\Big).$$

Example 4. Continued with Example 1. Let $\alpha = 0.6$, $\gamma = 0.4$, and $X = \{2,3\}$. Then $X^{\lessdot_{\alpha\gamma}} = (\{b\},\{a,d\})$, besides, $(\{b\},\{a,d\})^{\gtrdot_{\alpha\gamma}} = \{2\}\cup\{2\} = \{2\}$. Hence, $\big(\{2\},(\{b\},\{a,d\})\big)$ is a VPO3C. Remarkably, for the sake of simplicity, we clearly abbreviate $\big(\{2\},(\{b\},\{a,d\})\big)$ as $\big(2,(b,ad)\big)$.

From the above analysis, we can find that VPO3C has comprehensive and diverse information, which demonstrates that three-way decision has advantages in collecting information.

4. Two-way concept-cognitive learning based on VPO3C

In the earlier section, we have systematically discussed the relationship between the extent and the intent with respect to VPO3C. From cognitive granular, the consistency between objects and attributes can reflect the nature or the laws of things. If humans are interested in an unknown object, they usually have a vague and rough impression during their initial perception. The above uncertain impression is typically composed of some sufficient and necessary attributes. When this object aligns with its attributes, the systematic law of the human brain can be revealed.

In fact, in some practical applications, the establishment of concept $(X,(B,C))$ satisfying $X^{\lessdot_{\alpha\gamma}} = (B,C)$ and $(B,C)^{\gtrdot_{\alpha\gamma}} = X$ is not easy to directly obtain in a fuzzy formal context $(G,A,\tilde{R})$. Hence, how to find the three-way concepts from the given clues is an important problem of concept learning. Before beginning the discussion below, we refer to the clues (i.e., object information X and attribute information (B,C)) as information granules to illustrate the learning process of concept cognition.

4.1. Two-way concept-cognitive learning

Definition 8. Let $(G,A,\tilde{R})$ be a fuzzy formal context with $0 \le \gamma < \alpha \le 1$. $\lessdot_{\alpha\gamma}$ and $\gtrdot_{\alpha\gamma}$ are a pair of three-way learning operators. For any information granules $X \subseteq G,\ B,C \subseteq A$, we define

$$\mathcal{G}_1 = \{(X,(B,C))|(B,C) \subseteq X^{\lessdot_{\alpha\gamma}}, (B,C)^{\gtrdot_{\alpha\gamma}} \subseteq X\};$$
$$\mathcal{G}_2 = \{(X,(B,C))|(B,C) \supseteq X^{\lessdot_{\alpha\gamma}}, (B,C)^{\gtrdot_{\alpha\gamma}} \supseteq X\}.$$

If $(X,(B,C)) \in \mathcal{G}_1$, then $(X,(B,C))$ is a necessary $\alpha-\gamma$ three-way information granule. If $(X,(B,C)) \in \mathcal{G}_2$, then $(X,(B,C))$ is a sufficient $\alpha-\gamma$ three-way information granule. Simultaneously, if $(X,(B,C)) \in \mathcal{G}_1 \cap \mathcal{G}_2$, then $(X,(B,C))$ is a necessary and sufficient $\alpha-\gamma$ three-way information granule. Next, if $(X,(B,C)) \notin \mathcal{G}_1 \cup \mathcal{G}_2$, then $(X,(B,C))$ is an inconsistent $\alpha-\gamma$ three-way information granule.

As a matter of fact, a necessary and sufficient $\alpha-\gamma$ three-way information granule is a VPO3C. Due to the lack of VPO3Cs at the beginning of cognitive learning, the learning systems $\mathcal{G}_1$ and $\mathcal{G}_2$ are the inevitable requirement to learn three-way concepts from inconsistent $\alpha-\gamma$ three-way information granules.

Theorem 1. *Let $(G,A,\tilde{R})$ be a fuzzy formal context with $0 \le \gamma < \alpha \le 1$. $\mathcal{G}_1$ is a necessary $\alpha-\gamma$ three-way information granule set. For any $X \subseteq G,\ B,C \subseteq A$, then the following statements hold:*

1. $\left(X \cup (B,C)^{\gtrdot_{\alpha\gamma}}, X^{\lessdot_{\alpha\gamma}} \cup (B,C)\right) \in \mathcal{G}_1$;
2. $\left(X \cap (B,C)^{\gtrdot_{\alpha\gamma}}, X^{\lessdot_{\alpha\gamma}} \cap (B,C)\right) \in \mathcal{G}_1$;
3. $\left((B,C)^{\gtrdot_{\alpha\gamma}}, X^{\lessdot_{\alpha\gamma}} \cap (B,C)\right) \in \mathcal{G}_1$;
4. $\left(X \cup (B,C)^{\gtrdot_{\alpha\gamma}}, X^{\lessdot_{\alpha\gamma}}\right) \in \mathcal{G}_1$;
5. $\left(X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, X^{\lessdot_{\alpha\gamma}} \cap (B,C)\right) \in \mathcal{G}_1$;
6. $\left(X \cup (B,C)^{\gtrdot_{\alpha\gamma}}, (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}}\right) \in \mathcal{G}_1$.

Proof. We only prove items 1, 3 and 5 here. The proofs of items 2, 4 and 6 are similar to the previous three items.

1. It follows from Property 2.3 and 2.7 that $(X \cup (B,C)^{\gtrdot_{\alpha\gamma}})^{\lessdot_{\alpha\gamma}} \supseteq X^{\lessdot_{\alpha\gamma}} \cup (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}} \supseteq X^{\lessdot_{\alpha\gamma}} \cup (B,C)$. On the other hand, from Property 2.3 and 2.6, we have $(X^{\lessdot_{\alpha\gamma}} \cup (B,C))^{\gtrdot_{\alpha\gamma}} = X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}} \cup (B,C)^{\gtrdot_{\alpha\gamma}} \subseteq X \cup (B,C)^{\gtrdot_{\alpha\gamma}}$.

3. According to Property 2.3, we can get that $(B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}} \supseteq (B,C) \supseteq (B,C) \cap X^{\lessdot_{\alpha\gamma}}$. In addition, also from Property 2.7, it is evident that $(X^{\lessdot_{\alpha\gamma}} \cap (B,C))^{\gtrdot_{\alpha\gamma}} \subseteq X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}} \cap (B,C)^{\gtrdot_{\alpha\gamma}} \subseteq (B,C)^{\gtrdot_{\alpha\gamma}}$.

5. $X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}} = X^{\lessdot_{\alpha\gamma}} \supseteq X^{\lessdot_{\alpha\gamma}} \cap (B,C)$ holds naturally based on Property 2.4. Then we also obtain that $(X^{\lessdot_{\alpha\gamma}} \cap (B,C))^{\gtrdot_{\alpha\gamma}} \subseteq X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}} \cap (B,C)^{\gtrdot_{\alpha\gamma}} \subseteq X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}$. □

Theorem 2. *Let $(G, A, \tilde{R})$ be a fuzzy formal context with $0 \le \gamma < \alpha \le 1$. $\mathcal{G}_2$ is a sufficient $\alpha-\gamma$ three-way information granule set. For any $X \subseteq G$, $B, C \subseteq A$, then*

1. $\left(X \cap (B,C)^{\gtrdot_{\alpha\gamma}}, (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}}\right) \in \mathcal{G}_2$;
2. $\left(X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, X^{\lessdot_{\alpha\gamma}} \cup (B,C)\right) \in \mathcal{G}_2$.

Proof. For simplicity, we only prove item 1, and second one can be demonstrated in a similar way.

1. It is sure that $(X \cap (B,C)^{\gtrdot_{\alpha\gamma}})^{\lessdot_{\alpha\gamma}} = X^{\lessdot_{\alpha\gamma}} \cap (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}} \subseteq (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}}$ from Property 2.6. Furthermore, $(B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}} = (B,C)^{\gtrdot_{\alpha\gamma}} \supseteq X \cap (B,C)^{\gtrdot_{\alpha\gamma}}$. Thus $(X \cap (B,C)^{\gtrdot_{\alpha\gamma}}, (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}}) \in \mathcal{G}_2$. □

Theorems 1 and 2 show that machine can accurately learn several necessary $\alpha-\gamma$ three-way information granules and sufficient $\alpha-\gamma$ three-way information granules from the given general three-way information granules. Therefore, we also discuss the following theorems to illustrate the acquisition of necessary and sufficient $\alpha-\gamma$ three-way information granules.

Theorem 3. *Let $(G, A, \tilde{R})$ be a fuzzy formal context with $0 \le \gamma < \alpha \le 1$. $\mathcal{G}_1 \cap \mathcal{G}_2$ is a necessary and sufficient $\alpha-\gamma$ three-way information granule set. For an arbitrary $(X,(B,C)) \in \mathcal{G}_1$, then*

1. $\left((B,C)^{\gtrdot_{\alpha\gamma}} \cap X, \left((B,C)^{\gtrdot_{\alpha\gamma}} \cap X\right)^{\lessdot_{\alpha\gamma}}\right) \in \mathcal{G}_1 \cap \mathcal{G}_2$;
2. $\left(\left((B,C) \cup X^{\lessdot_{\alpha\gamma}}\right)^{\gtrdot_{\alpha\gamma}}, (B,C) \cup X^{\lessdot_{\alpha\gamma}}\right) \in \mathcal{G}_1 \cap \mathcal{G}_2$.

Proof. We only prove item 1. Then by the similar proof we can demonstrate second one.

1. Because $(X,(B,C)) \in \mathcal{G}_1$, then $(B,C) \subseteq X^{\lessdot_{\alpha\gamma}}$ and $(B,C)^{\gtrdot_{\alpha\gamma}} \subseteq X$, which implies that $(B,C)^{\gtrdot_{\alpha\gamma}} \cap X = (B,C)^{\gtrdot_{\alpha\gamma}}$. Next, we obtain $((B,C)^{\gtrdot_{\alpha\gamma}} \cap X)^{\lessdot_{\alpha\gamma}} = (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}} = ((B,C)^{\gtrdot_{\alpha\gamma}} \cap X)^{\lessdot_{\alpha\gamma}}$ and $((B,C)^{\gtrdot_{\alpha\gamma}} \cap X)^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}} = (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}} = (B,C)^{\gtrdot_{\alpha\gamma}} = (B,C)^{\gtrdot_{\alpha\gamma}} \cap X$. Therefore, $((B,C)^{\gtrdot_{\alpha\gamma}} \cap X, ((B,C)^{\gtrdot_{\alpha\gamma}} \cap X)^{\lessdot_{\alpha\gamma}}) \in \mathcal{G}_1 \cap \mathcal{G}_2$. □

Theorem 4. *Let $(G, A, \tilde{R})$ be a fuzzy formal context with $0 \le \gamma < \alpha \le 1$. $\mathcal{G}_1 \cap \mathcal{G}_2$ is a necessary and sufficient $\alpha-\gamma$ three-way information granule set. For an arbitrary $(X,(B,C)) \in \mathcal{G}_2$, then the following statements hold:*

1. $\left((B,C)^{\gtrdot_{\alpha\gamma}} \cup X, \left((B,C)^{\gtrdot_{\alpha\gamma}} \cup X\right)^{\lessdot_{\alpha\gamma}}\right) \in \mathcal{G}_1 \cap \mathcal{G}_2$;
2. $\left(\left((B,C) \cap X^{\lessdot_{\alpha\gamma}}\right)^{\gtrdot_{\alpha\gamma}}, (B,C) \cap X^{\lessdot_{\alpha\gamma}}\right) \in \mathcal{G}_1 \cap \mathcal{G}_2$.

Proof. We only prove item 1 here.

1. According to $(X,(B,C)) \in \mathcal{G}_2$, there holds $(B,C) \supseteq X^{\lessdot_{\alpha\gamma}}$ and $(B,C)^{\gtrdot_{\alpha\gamma}} \supseteq X$, which means that $(B,C)^{\gtrdot_{\alpha\gamma}} \cup X = (B,C)^{\gtrdot_{\alpha\gamma}}$. Then we obtain $((B,C)^{\gtrdot_{\alpha\gamma}} \cup X)^{\lessdot_{\alpha\gamma}} = (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}} = ((B,C)^{\gtrdot_{\alpha\gamma}} \cup X)^{\lessdot_{\alpha\gamma}}$, and $((B,C)^{\gtrdot_{\alpha\gamma}} \cup X)^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}} = (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}} = (B,C)^{\gtrdot_{\alpha\gamma}} = (B,C)^{\gtrdot_{\alpha\gamma}} \cup X$. Therefore, $((B,C)^{\gtrdot_{\alpha\gamma}} \cup X, ((B,C)^{\gtrdot_{\alpha\gamma}} \cup X)^{\lessdot_{\alpha\gamma}}) \in \mathcal{G}_1 \cap \mathcal{G}_2$. □

Theorems 3 and 4 respectively illustrate that there are two ways to learn necessary and sufficient $\alpha-\gamma$ three-way information granules from necessary or sufficient $\alpha-\gamma$ three-way information granule.

Based on above analysis, we conclude that there exist 16 ways to learn and transform an inconsistent information granule into necessary and sufficient $\alpha-\gamma$ three-way information granules, and its explanation is depicted in Fig. 2.

4.2. A concise learning mechanism based on information granules

In subsection 3.2, in a given fuzzy formal context, for any information granules $X \subseteq G$, $B, C \subseteq A$, then we see that $(X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, X^{\lessdot_{\alpha\gamma}})$ and $((B,C)^{\gtrdot_{\alpha\gamma}}, (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}})$ are VPO3Cs of $\mathcal{G}_1 \cap \mathcal{G}_2$. In fact, if $(X,(B,C)) \in \mathcal{G}_1 \cap \mathcal{G}_2$, then $(X,(B,C)) = (X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, X^{\lessdot_{\alpha\gamma}}) =$

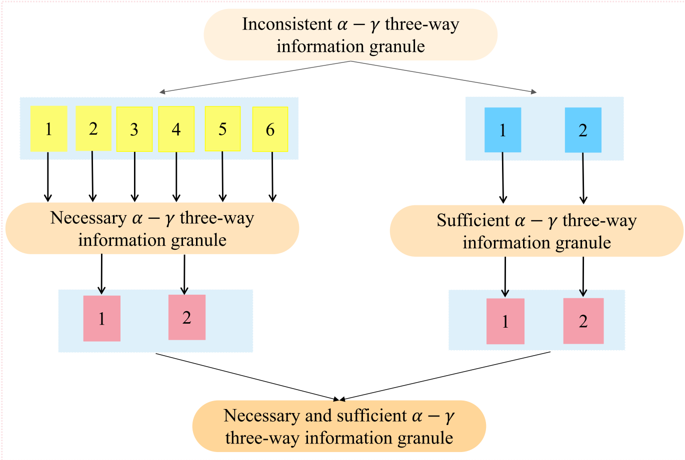

Fig. 2. The learning process of necessary and sufficient $\alpha-\gamma$ three-way information granules.

$((B,C)^{\gtrdot_{\alpha\gamma}}, (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}})$. From Theorems 1 and 2, we can find that whether it is a necessary $\alpha-\gamma$ three-way information granule or a sufficient $\alpha-\gamma$ three-way information granule, the final VPO3C can be transformed into $(X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, X^{\lessdot_{\alpha\gamma}})$ and $((B,C)^{\gtrdot_{\alpha\gamma}}, (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}})$. In such case, the extents and intents of necessary $\alpha-\gamma$ three-way information granule can be considered as a special general three-way information granule. Similarly, the sufficient $\alpha-\gamma$ three-way information granule also has the same situation. Thus, assume that Y^F and $(B,C)^H$ represent the extent and intent of general three-way information granules. Then from Theorems 1 and 2, we can obtain $Y^F = \{X \cup (B,C)^{\gtrdot_{\alpha\gamma}}, X \cap (B,C)^{\gtrdot_{\alpha\gamma}}, X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, (B,C)^{\gtrdot_{\alpha\gamma}}\}$ and $(B,C)^H = \{X^{\lessdot_{\alpha\gamma}} \cup (B,C), X^{\lessdot_{\alpha\gamma}} \cap (B,C), X^{\lessdot_{\alpha\gamma}}, (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}}\}$. Therefore, $\mathcal{G}_1 \cap \mathcal{G}_2 = \{(Y^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, Y^{\lessdot_{\alpha\gamma}}) | Y \in Y^F\} \cup \{((E,F)^{\gtrdot_{\alpha\gamma}}, (E,F)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}}) | (E,F) \in (B,C)^H\}$.

Theorem 5. *Let $(G, A, \tilde{R})$ be a fuzzy formal context with $0 \le \gamma < \alpha \le 1$. $\mathcal{G}_1 \cap \mathcal{G}_2$ is a necessary and sufficient $\alpha-\gamma$ three-way information granule set, then the following statements hold:*

1. $\left(\left(X \cup (B,C)^{\gtrdot_{\alpha\gamma}}\right)^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, \left(X \cup (B,C)^{\gtrdot_{\alpha\gamma}}\right)^{\lessdot_{\alpha\gamma}}\right) \in \mathcal{G}_1 \cap \mathcal{G}_2$;
2. $\left(\left(X \cap (B,C)^{\gtrdot_{\alpha\gamma}}\right)^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, \left(X \cap (B,C)^{\gtrdot_{\alpha\gamma}}\right)^{\lessdot_{\alpha\gamma}}\right) \in \mathcal{G}_1 \cap \mathcal{G}_2$;
3. $\left(X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, X^{\lessdot_{\alpha\gamma}}\right) \in \mathcal{G}_1 \cap \mathcal{G}_2$;
4. $\left((B,C)^{\gtrdot_{\alpha\gamma}}, (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}}\right) \in \mathcal{G}_1 \cap \mathcal{G}_2$;
5. $\left(\left(X^{\lessdot_{\alpha\gamma}} \cup (B,C)\right)^{\gtrdot_{\alpha\gamma}}, \left(X^{\lessdot_{\alpha\gamma}} \cup (B,C)\right)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}}\right) \in \mathcal{G}_1 \cap \mathcal{G}_2$;
6. $\left(\left(X^{\lessdot_{\alpha\gamma}} \cap (B,C)\right)^{\gtrdot_{\alpha\gamma}}, \left(X^{\lessdot_{\alpha\gamma}} \cap (B,C)\right)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}}\right) \in \mathcal{G}_1 \cap \mathcal{G}_2$;

Proof. It follows immediately from Theorems 1, 2, 3 and 4. □

Theorem 5 shows that there exist 6 ways to directly learn the VPO3Cs. However, these three-way concepts might be the same, so the number is less than or equal to six. Its explanation is described in Fig. 3.

To better explain the construction process of VPO3C in a fuzzy formal context, the following example is used to supplement the calculation steps. Moreover, Algorithm 1 presents the corresponding concept formation algorithm, which can fully excavate rich knowledge.

Example 5. Continued with Example 1. Let $\alpha = 0.6$, $\gamma = 0.4$, $X = \{12\}$, and $(B,C) = (be, a)$. Then using Eq. (1) we have $X^{\lessdot_{\alpha\gamma}} = (bf, ad)$ and $X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}} = \{12\}$. At the same time, $(B,C)^{\gtrdot_{\alpha\gamma}} = \{24\}$ and $(B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}} = (be, ad)$. Hence, VPO3Cs are shown as follows:

1. $\left(\left(X \cup (B,C)^{\gtrdot_{\alpha\gamma}}\right)^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, \left(X \cup (B,C)^{\gtrdot_{\alpha\gamma}}\right)^{\lessdot_{\alpha\gamma}}\right) = \left(124, (bef, ad)\right)$;
2. $\left(\left(X \cap (B,C)^{\gtrdot_{\alpha\gamma}}\right)^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, \left(X \cap (B,C)^{\gtrdot_{\alpha\gamma}}\right)^{\lessdot_{\alpha\gamma}}\right) = \left(2, (b, ad)\right)$;
3. $\left(X^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, X^{\lessdot_{\alpha\gamma}}\right) = \left(12, (bf, ad)\right)$;
4. $\left((B,C)^{\gtrdot_{\alpha\gamma}}, (B,C)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}}\right) = \left(24, (be, ad)\right)$;
5. $\left(\left(X^{\lessdot_{\alpha\gamma}} \cup (B,C)\right)^{\gtrdot_{\alpha\gamma}}, \left(X^{\lessdot_{\alpha\gamma}} \cup (B,C)\right)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}}\right) = \left(124, (bef, ad)\right)$;
6. $\left(\left(X^{\lessdot_{\alpha\gamma}} \cap (B,C)\right)^{\gtrdot_{\alpha\gamma}}, \left(X^{\lessdot_{\alpha\gamma}} \cap (B,C)\right)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}}\right) = \left(2, (b, ad)\right)$.

As it can be seen from Example 5, four of VPO3Cs are different.

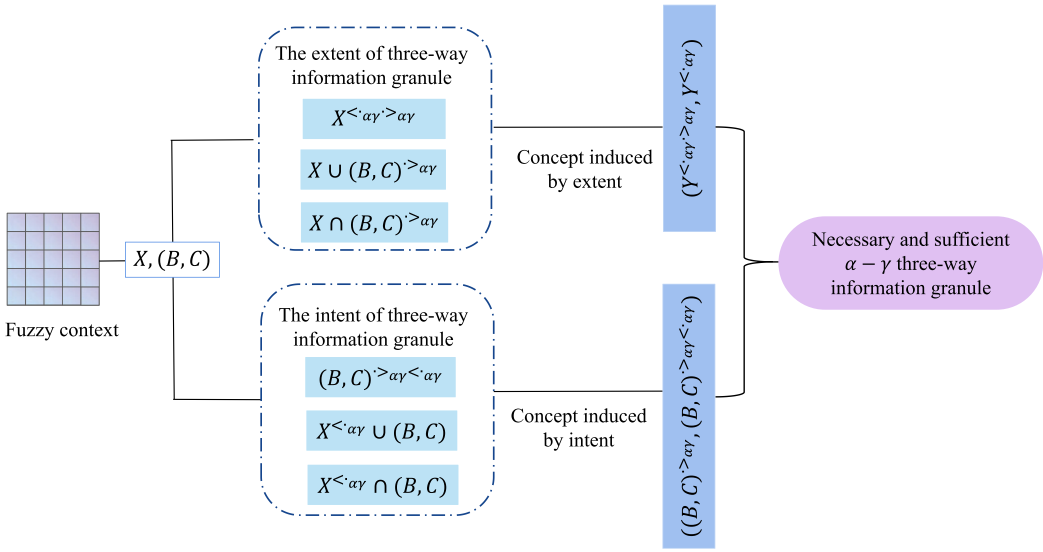

Fig. 3. The concise learning process of necessary and sufficient $\alpha - \gamma$ three-way information granules.

Algorithm 1: Learning VPO3Cs (LVPTC).

Input: A fuzzy formal context $(G, A, \tilde{R})$, α, γ, and given three-way information granules X and (B, C).
Output: VPO3C set $\mathcal{G}_1 \cap \mathcal{G}_2$.

Initialize: $\mathcal{G}_1 \cap \mathcal{G}_2 \leftarrow \emptyset$.
if $(X, (B, C)) \notin \mathcal{G}_1 \cup \mathcal{G}_2$ **then**
  Compute the extent and intent of three-way information granules Y^F and $(B, C)^H$;
  for each $Y \in Y^F$ **do**
    Compute the VPO3C $(Y^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, Y^{\lessdot_{\alpha\gamma}})$ from Theorem 5;
  end for
  for each $(E, F) \in (B, C)^H$ **do**
    Compute the VPO3C $\left((E, F)^{\gtrdot_{\alpha\gamma}}, (E, F)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}}\right)$ from Theorem 5;
  end for
end if
$\mathcal{G}_1 \cap \mathcal{G}_2 \leftarrow (Y^{\lessdot_{\alpha\gamma}\gtrdot_{\alpha\gamma}}, Y^{\lessdot_{\alpha\gamma}})$ and $\mathcal{G}_1 \cap \mathcal{G}_2 \leftarrow ((E, F)^{\gtrdot_{\alpha\gamma}}, (E, F)^{\gtrdot_{\alpha\gamma}\lessdot_{\alpha\gamma}})$.
Return $\mathcal{G}_1 \cap \mathcal{G}_2$.

5. Updating mechanism of VPO3Cs based on information granules

With the viewpoint of cognitive computing, the object set X and the pair of attributes subset (B, C) in the information granules will be updated as time goes by, which shows that the previous three-way concepts should be updated to simulate intelligence behaviors of the human brain. As it can be seen from Example 1, five patients with heart disease have six symptoms. As time progresses, there will be more patients with other symptoms, such as abdominal distention, jaundice, anxiety and so forth. In this case, therefore, it is desirable to update the VPO3Cs when facing the scenario of dynamic information granules. To tackle this issue, we mainly focus on two dynamic updating mechanisms of information granules. The first is the dynamic information on the object set X, and the other one is the dynamic information on the pair of attribute (B, C).

5.1. Updating mechanism based on VPO3Cs

In this subsection, we will introduce some symbolic descriptions before stating our issue.

Assume that n object sets $G_1, G_2, \cdots, G_n$ with $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n$ are denoted as $\{G_t\}^{\uparrow}$. Then n attribute sets $A_1, A_2, \cdots, A_n$ with $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$ are denoted as $\{A_t\}^{\uparrow}$. Furthermore, n object sets in the information granule $X_1, X_2, \cdots, X_n$ with $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n$ are denoted as $\{X_t\}^{\uparrow}$. The n attribute sets in the information granule $(B, C)_1, (B, C)_2, \cdots, (B, C)_n$ with $(B, C)_1 \subseteq (B, C)_2 \subseteq \cdots \subseteq (B, C)_n$ are denoted as $\{(B, C)_t\}^{\uparrow}$. Hereafter, for any $i \leq n$, we denote by $\Gamma(A_i)$ the power set of the pair of attribute set A_i.

Definition 9. Let $(G, A, \tilde{R})$ be a fuzzy formal context with $0 \leq \gamma < \alpha \leq 1$. G_{i-1}, G_i are object sets of $\{G_t\}^{\uparrow}$ and A_{i-1}, A_i are attribute sets of $\{A_t\}^{\uparrow}$. X_{i-1}, X_i are the object sets in the information granule of $\{X_t\}^{\uparrow}$. $(B, C)_{i-1}, (B, C)_i$ are the attribute sets in the information granule of $\{(B, C)_t\}^{\uparrow}$. Denote $\triangle G_{i-1} = G_i - G_{i-1}$, $\triangle A_{i-1} = A_i - A_{i-1}$, $\triangle X_{i-1} = X_i - X_{i-1}$ and $\triangle (B, C)_{i-1} = (B, C)_i - (B, C)_{i-1}$. Assume that

$(1)F_{i-1}: 2^{X_{i-1}} \to \Gamma(A_{i-1}),\ \ H_{i-1}: 2^{(B,C)_{i-1}} \to 2^{G_{i-1}},$

$(2)F_{\triangle X_{i-1}}: 2^{\triangle X_{i-1}} \to \Gamma(A_{i-1}),\ H_{\triangle X_{i-1}}: 2^{(B,C)_{i-1}} \to 2^{\triangle G_{i-1}},$

$(3)F_{\triangle(B,C)_{i-1}}: 2^{X_i} \to \Gamma(\triangle A_{i-1}),\ H_{\triangle(B,C)_{i-1}}: 2^{\triangle(B,C)_{i-1}} \to 2^{G_i},$

$(4)F_i: 2^{X_i} \to \Gamma(A_i),\quad H_{i-1}: 2^{(B,C)_i} \to 2^{G_i},$

are four pairs of cognitive mappings which satisfy the following properties:

$$F_i(X_i) = F_{i-1}(X_{i-1}) \cap F_{\triangle X_{i-1}}(\triangle X_{i-1}) \cup F_{\triangle(B,C)_{i-1}}(X_i),\quad \triangle X_{i-1} \neq \emptyset, \tag{2}$$

$$H_i((B,C)_i) = H_{i-1}\big((B,C)_{i-1}\big) \cup H_{\triangle X_{i-1}}\big((B,C)_{i-1}\big) \cup H_{\triangle(B,C)_{i-1}}\big(\triangle(B,C)_{i-1}\big),\ \ \triangle(B,C)_{i-1} \neq \emptyset, \tag{3}$$

where $H_{\triangle(B,C)_{i-1}}\big(\triangle(B,C)_{i-1}\big)$ is set to be empty when $\triangle(B,C)_{i-1} = \emptyset$, while $F_{\triangle X_{i-1}}(\triangle X_{i-1})$ is not necessarily empty when $\triangle X_{i-1} = \emptyset$.

From the above statement, F_i, H_i can be regarded as the current state of F_{i-1}, H_{i-1} with the newly updated information $F_{\triangle X_{i-1}}$, $H_{\triangle X_{i-1}}$ and $F_{\triangle(B,C)_{i-1}}$, $H_{\triangle(B,C)_{i-1}}$. Therefore, from F_{i-1}, H_{i-1} to F_i, H_i, it is necessary to update three-way concepts. It should be pointed out that this cognitive process is considered as transformation between information granules from the perspective of granular computing. In addition, it is important to update the necessary and sufficient $\alpha-\gamma$ three-way information granules $(\mathcal{G}_1 \cap \mathcal{G}_2)^i$ with the combination of $(\mathcal{G}_1 \cap \mathcal{G}_2)^1, (\mathcal{G}_1 \cap \mathcal{G}_2)^2, \cdots, (\mathcal{G}_1 \cap \mathcal{G}_2)^n$. To address the above-state problems, we will continue to discuss the dynamic mechanism when the objects and the attributes are added to the information granules, respectively.

5.2. Updating mechanism when the objects in the information granules are added

This subsection primarily focuses on the approach for dynamically updating the VPO3Cs when the information X in the object set will be increased and the information (B,C) in the attribute set will be unchanged owing to the adding of numerous objects and attributes in the continuous data set. From the above discussion, it should be noted that the key to obtaining the VPO3Cs is divided into two main components. First, it is evident that $X^I \in \{X, X \cap (B,C)^{\geqslant_{\alpha\gamma}}, X \cup (B,C)^{\geqslant_{\alpha\gamma}}\}$ and $(B,C)^I \in \{(B,C), (B,C) \cap X^{\lessdot_{\alpha\gamma}}, (B,C) \cup X^{\lessdot_{\alpha\gamma}}\}$ could be considered as derived information granules induced by the initial information granules X and (B,C). Furthermore, based on derived information granules X^I and $(B,C)^I$, we can construct the extents Y^F and intents $(B,C)^H$ of VPO3Cs and thereby implement updating efficiency. Therefore, we propose the updating mechanism when the object set in the information granule is added.

When the information X in the object set is added and the information (B,C) in the attribute set is unchanged, the above three derived information granules $X_i, X_i \cap (B,C)_i^{\geqslant_{\alpha\gamma}}, X_i \cup (B,C)_i^{\geqslant_{\alpha\gamma}}$ will be changed simultaneously, where i represents the i-th state. For convenience, we first assume that $X^I = X_i$ and $(B,C)^I = (B,C)_i$ are information granules on the object and attribute sets. Next, we investigate the updating mechanism with respect to the VPO3Cs.

Proposition 1. *Let $(G, A, \tilde{R})$ be a fuzzy formal context with $0 \le \gamma < \alpha \le 1$, $\triangle X_{i-1} \neq \emptyset$ and $\triangle(B,C)_{i-1} = \emptyset$. Suppose $X^I = X_i$ and $(B,C)^I = (B,C)_i$, then we have*

1. If $Y^F = X^I$, then $\triangle Y^F_{i-1} = X_i - X_{i-1}$,

2. If $Y^F = X^I \cup H_i\big((B,C)^I\big)$, then $Y^F = X_{i-1} \cup \triangle X_{i-1} \cup H_{i-1}\big((B,C)_{i-1}\big) \cup H_{\triangle X_{i-1}}\big((B,C)_{i-1}\big)$. Command $Y^F_{i-1} = X_{i-1} \cup H_{i-1}\big((B,C)_{i-1}\big)$ and $\triangle Y^F_{i-1} = \triangle X_{i-1} \cup H_{\triangle X_{i-1}}\big((B,C)_{i-1}\big)$,

3. If $Y^F = X^I \cap H_i\big((B,C)^I\big)$, then $Y^F = X_{i-1} \cup \triangle X_{i-1} \cap \big(H_{i-1}((B,C)_{i-1}) \cup H_{\triangle X_{i-1}}((B,C)_{i-1})\big)$. Command $Y^F_{i-1} = X_{i-1} \cap H_{i-1}\big((B,C)_{i-1}\big)$ and $\triangle Y^F_{i-1} = \triangle X_{i-1} \cap H_{\triangle X_{i-1}}\big((B,C)_{i-1}\big)$.

Then VPO3Cs are updated as follows:

$$\begin{aligned}\big(H_i F_i(Y^F), F_i(Y^F)\big) &= \Big(H_i\big(F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1}) \cup F_{\triangle(B,C)_{i-1}}(Y^F)\big), F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1}) \cup F_{\triangle(B,C)_{i-1}}(Y^F)\Big)\\ &= \Big(H_{i-1}\big(F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1})\big) \cup H_{\triangle Y^F_{i-1}}\big(F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1})\big) \cup H_{\triangle(B,C)_{i-1}}\big(F_{\triangle(B,C)_{i-1}}(Y^F)\big),\\ &\quad F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1}) \cup F_{\triangle(B,C)_{i-1}}(Y^F)\Big).\end{aligned} \tag{4}$$

4. If $(B,C)^H = (B,C)^I$, then $\triangle(B,C)^H_{i-1} = \emptyset$,

5. If $(B,C)^H = F_i(X^I) \cap (B,C)^I$, then $(B,C)^H = F_{i-1}(X_{i-1}) \cap F_{\triangle X_{i-1}}(\triangle X_{i-1}) \cup F_{\triangle(B,C)_{i-1}}(X_i) \cap (B,C)_i$. Command $(B,C)^H_{i-1} = F_{i-1}(X_{i-1}) \cap F_{\triangle X_{i-1}}(\triangle X_{i-1}) \cap (B,C)_i$ and $\triangle(B,C)^H_{i-1} = F_{\triangle(B,C)_{i-1}}(X_i) \cap (B,C)_i = \emptyset$.

Thus VPO3Cs are updated as follows:

$$\begin{aligned}&\Big(H_i\big((B,C)^H\big), F_i H_i\big((B,C)^H\big)\Big) = \Big(H_{i-1}\big((B,C)^H_{i-1}\big) \cup H_{\triangle X_{i-1}}\big((B,C)^H_{i-1}\big),\\ &F_{i-1}H_{i-1}\big((B,C)^H_{i-1}\big) \cap F_{\triangle X_{i-1}}H_{\triangle X_{i-1}}\big((B,C)^H_{i-1}\big) \cup F_{\triangle(B,C)^H_{i-1}}\Big(H_{i-1}\big((B,C)^H_{i-1}\big) \cup H_{\triangle X_{i-1}}\big((B,C)^H_{i-1}\big)\Big)\Big).\end{aligned} \tag{5}$$

6. If $(B,C)^H = F_i(X^I) \cup (B,C)^I$, then $(B,C)^H = F_{i-1}(X_{i-1}) \cap F_{\triangle X_{i-1}}(\triangle X_{i-1}) \cup F_{\triangle (B,C)_{i-1}}(X_i) \cup (B,C)_i$. Command $(B,C)^H_{i-1} = F_{i-1}(X_{i-1}) \cap F_{\triangle X_{i-1}}(\triangle X_{i-1}) \cup (B,C)_i$ and $\triangle (B,C)^H_{i-1} = F_{\triangle (B,C)_{i-1}}(X_i)$.

Then VPO3C is updated as follows:

$$\begin{aligned}&\Big(H_i\big((B,C)^H\big), F_iH_i\big((B,C)^H\big)\Big) = \Big(H_{i-1}\big((B,C)^H_{i-1}\big) \cup H_{\triangle X_{i-1}}\big((B,C)^H_{i-1}\big) \cup H_{\triangle (B,C)^H_{i-1}}\big(\triangle (B,C)^H_{i-1}\big),\\&F_{i-1}\Big(H_{i-1}\big((B,C)^H_{i-1}\big) \cup \Big(H_{\triangle (B,C)^H_{i-1}}\big(\triangle (B,C)^H_{i-1}\big) \cap G_{i-1}\Big)\Big) \cap F_{\triangle X_{i-1}}\Big(H_{\triangle X_{i-1}}\big((B,C)^H_{i-1}\big) \cup \Big(H_{\triangle (B,C)^H_{i-1}}\big(\triangle (B,C)^H_{i-1}\big)\\&\cap \triangle G_{i-1}\Big)\Big) \cup F_{\triangle (B,C)^H_{i-1}}\Big(H_{i-1}\big((B,C)^H_{i-1}\big) \cup H_{\triangle X_{i-1}}\big((B,C)^H_{i-1}\big) \cup H_{\triangle (B,C)^H_{i-1}}\big(\triangle (B,C)^H_{i-1}\big)\Big)\Big).\end{aligned} \tag{6}$$

Proof. We only prove Eq. (4) here, and Eq. (5) and Eq. (6) can be demonstrated in a similar way. According to Definition 9, we can get $F_i(Y^F_i) = F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1}) \cup F_{\triangle (B,C)_{i-1}}(Y^F)$. For $(a_1,b_1),(a_2,b_2),(a_3,b_3) \in F_i(Y^F_i)$, then assuming that $(a_1,b_1),(a_2,b_2) \in F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1})$ and $(a_3,b_3) \in F_{\triangle (B,C)_{i-1}}(Y^F)$. From Property 2, we conclude

$$\begin{aligned}&H_i\big((a_1a_2a_3, b_1b_2b_3)\big) = H_i\big((a_1,b_1)\big) \cup H_i\big((a_2,b_2)\big) \cup H_i\big((a_3,b_3)\big)\\&= \Big(H_{i-1}\big((a_1,b_1)\big) \cup H_{\triangle Y^F_{i-1}}\big((a_1,b_1)\big)\Big) \cup \Big(H_{i-1}\big((a_2,b_2)\big) \cup H_{\triangle Y^F_{i-1}}\big((a_2,b_2)\big)\Big) \cup H_{\triangle (B,C)_{i-1}}\big((a_3,b_3)\big)\\&= \Big(H_{i-1}\big((a_1,b_1)\big) \cup H_{i-1}\big((a_2,b_2)\big)\Big) \cup \Big(H_{\triangle Y^F_{i-1}}\big((a_1,b_1)\big) \cup H_{\triangle Y^F_{i-1}}\big((a_2,b_2)\big)\Big) \cup H_{\triangle (B,C)_{i-1}}\big((a_3,b_3)\big)\\&= H_{i-1}\big((a_1a_2, b_1b_2)\big) \cup H_{\triangle Y^F_{i-1}}\big((a_1a_2, b_1b_2)\big) \cup H_{\triangle (B,C)_{i-1}}\big((a_3,b_3)\big).\end{aligned}$$

Finally, it follows that Eq. (4) holds by using recursive approach. □

As shown in Proposition 1, we know that there are six methods of updating the VPO3Cs when $X^I = X_i$ and $(B,C)^I = (B,C)_i$ from the current state F_{i-1}, H_{i-1} to F_i, H_i with the newly input information on the object set and attribute set. Hereafter, in fact, assuming $X^I = X_i \cap H_i\big((B,C)_i\big)$ and $(B,C)^I = (B,C)_i$, the VPO3Cs generated will be partially identical to those generated when $X^I = X_i$ and $(B,C)^I = (B,C)_i$. Thus, in the following proposition, we only discuss the case where the concepts are different from Proposition 1.

Proposition 2. *Let $(G, A, \tilde{R})$ be a fuzzy formal context with $0 \le \gamma < \alpha \le 1$, $\triangle X_{i-1} \ne \emptyset$ and $\triangle (B,C)_{i-1} = \emptyset$. Suppose $X^I = X_i \cap H_i\big((B,C)_i\big)$ and $(B,C)^I = (B,C)_i$, then we have $X^I_{i-1} = X_{i-1} \cap H_{i-1}\big((B,C)_{i-1}\big)$ and $\triangle X^I_{i-1} = \triangle X_{i-1} \cap H_{\triangle X_{i-1}}\big((B,C)_{i-1}\big)$. The following statement holds:*

1. If $(B,C)^H = F_i(X^I) \cup (B,C)^I$, then $(B,C)^H = F_{i-1}(X^I_{i-1}) \cap F_{\triangle X^I_{i-1}}(\triangle X^I_{i-1}) \cup F_{\triangle (B,C)_{i-1}}(X^I_i) \cup (B,C)_i$. Command $(B,C)^H_{i-1} = F_{i-1}(X^I_{i-1}) \cap F_{\triangle X^I_{i-1}}(\triangle X^I_{i-1}) \cup (B,C)_i$ and $\triangle (B,C)^H_{i-1} = F_{\triangle (B,C)_{i-1}}(X^I_i)$.

Then VPO3C is updated as follows:

$$\begin{aligned}&\Big(H_i\big((B,C)^H\big), F_iH_i\big((B,C)^H\big)\Big) = \Big(H_{i-1}\big((B,C)^H_{i-1}\big) \cup H_{\triangle X_{i-1}}\big((B,C)^H_{i-1}\big) \cup H_{\triangle (B,C)^H_{i-1}}\big(\triangle (B,C)^H_{i-1}\big),\\&F_{i-1}\Big(H_{i-1}\big((B,C)^H_{i-1}\big) \cup \Big(H_{\triangle (B,C)^H_{i-1}}\big(\triangle (B,C)^H_{i-1}\big) \cap G_{i-1}\Big)\Big) \cap F_{\triangle X_{i-1}}\Big(H_{\triangle X_{i-1}}\big((B,C)^H_{i-1}\big) \cup \Big(H_{\triangle (B,C)^H_{i-1}}\big(\triangle (B,C)^H_{i-1}\big) \cap \triangle G_{i-1}\Big)\Big)\\&\cup F_{\triangle (B,C)^H_{i-1}}\Big(H_{i-1}\big((B,C)^H_{i-1}\big) \cup H_{\triangle X_{i-1}}\big((B,C)^H_{i-1}\big) \cup H_{\triangle (B,C)^H_{i-1}}\big(\triangle (B,C)^H_{i-1}\big)\Big)\Big).\end{aligned} \tag{7}$$

Similarly, VPO3Cs generated when $X^I = X_i \cup H_i\big((B,C)_i\big)$ and $(B,C)^I = (B,C)_i$ are partially the same as those generated when $X^I = X_i$ and $(B,C)^I = (B,C)_i$. Hence, two different approaches to updating three-way concepts are proposed in Proposition 3.

Proposition 3. *Let $(G, A, \tilde{R})$ be a fuzzy formal context with $0 \le \gamma < \alpha \le 1$, $\triangle X_{i-1} \ne \emptyset$ and $\triangle (B,C)_{i-1} = \emptyset$. Suppose $X^I = X_i \cup H_i\big((B,C)_i\big)$ and $(B,C)^I = (B,C)_i$, then we obtain $X^I_{i-1} = X_{i-1} \cup H_{i-1}\big((B,C)_{i-1}\big)$ and $\triangle X^I_{i-1} = \triangle X_{i-1} \cup H_{\triangle X_{i-1}}\big((B,C)_{i-1}\big)$. The following propositions hold:*

1. If $(B,C)^H = F_i(X^I) \cap (B,C)^I$, then $(B,C)^H = F_i\Big(X_i \cup H_i\big((B,C)_i\big)\Big) \cap (B,C)^I$. Command $(B,C)^H_{i-1} = (B,C)_{i-1} \cap F_{i-1}\Big(X_{i-1} \cup H_{i-1}\big((B,C)_{i-1}\big)\Big) \cap F_{\triangle X^I_{i-1}}\Big(\triangle X_{i-1} \cup H_{\triangle X_{i-1}}\big((B,C)_{i-1}\big)\Big)$ and $\triangle (B,C)^H_{i-1} = \emptyset$.

Then VPO3C is updated as follows:

$$\begin{aligned}&\Big(H_i\big((B,C)^H\big), F_iH_i\big((B,C)^H\big)\Big) = \Big(H_{i-1}\big((B,C)^H_{i-1}\big) \cup H_{\triangle X_{i-1}}\big((B,C)^H_{i-1}\big),\\&F_{i-1}H_{i-1}\big((B,C)^H_{i-1}\big) \cap F_{\triangle X_{i-1}}H_{\triangle X_{i-1}}\big((B,C)^H_{i-1}\big) \cup F_{\triangle (B,C)^H_{i-1}}\Big(H_{i-1}\big((B,C)^H_{i-1}\big) \cup H_{\triangle X_{i-1}}\big((B,C)^H_{i-1}\big)\Big)\Big).\end{aligned} \tag{8}$$

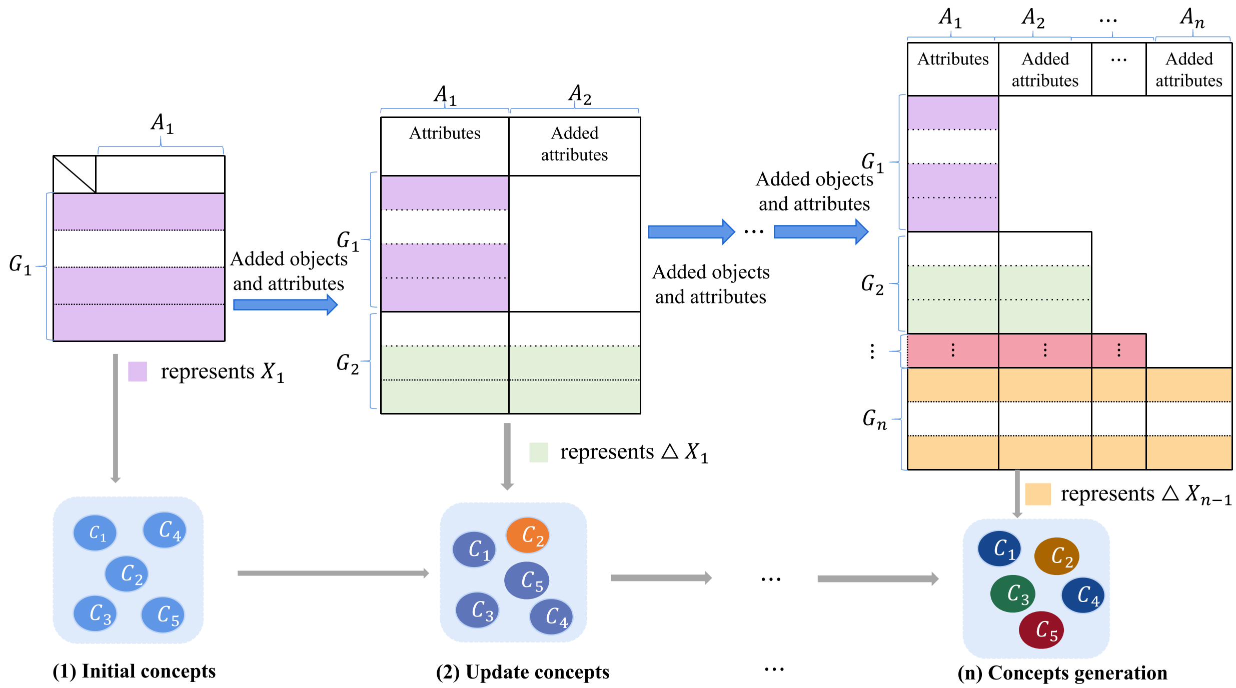

Fig. 4. The dynamic learning process of $\alpha-\gamma$ three-way concepts when the objects in the information granules are added.

2. If $(B,C)^H = F_i(X^I)\cup(B,C)^I$, command $(B,C)^H_{i-1} = (B,C)_{i-1}\cup F_{i-1}\left(X_{i-1}\cup H_{i-1}((B,C)_{i-1})\right)\cap F_{\triangle X^I_{i-1}}\left(\triangle X_{i-1}\cup H_{\triangle X_{i-1}}((B,C)_{i-1})\right)$ and $\triangle(B,C)^H_{i-1} = F_{\triangle(B,C)_{i-1}}(X^I)$.

Thus VPO3C is updated as follows:

$$\begin{aligned}&\left(H_i((B,C)^H), F_iH_i((B,C)^H)\right) = \Big(H_{i-1}((B,C)^H_{i-1}) \cup H_{\triangle X_{i-1}}((B,C)^H_{i-1}) \cup H_{\triangle(B,C)^H_{i-1}}(\triangle(B,C)^H_{i-1}),\\ &F_{i-1}\Big(H_{i-1}((B,C)^H_{i-1}) \cup \left(H_{\triangle(B,C)^H_{i-1}}(\triangle(B,C)^H_{i-1}) \cap G_{i-1}\right)\Big) \cap F_{\triangle X_{i-1}}\Big(H_{\triangle X_{i-1}}((B,C)^H_{i-1}) \cup \Big(H_{\triangle(B,C)^H_{i-1}}(\triangle(B,C)^H_{i-1})\\ &\cap \triangle G_{i-1}\Big)\Big) \cup F_{\triangle(B,C)^H_{i-1}}\left(H_{i-1}((B,C)^H_{i-1}) \cup H_{\triangle X_{i-1}}((B,C)^H_{i-1}) \cup H_{\triangle(B,C)^H_{i-1}}(\triangle(B,C)^H_{i-1})\right)\Big).\end{aligned} \tag{9}$$

From Propositions 1-3, one can conclude that there exist nine methods of updating VPO3Cs from the current state F_{i-1}, H_{i-1} to F_i, H_i when the information X on the object set is added and the information (B,C) on the attribute set is unchanged. Fig. 4 vividly shows the dynamic concept learning process by recursive approach based on VPO3Cs.

From this point of view, the corresponding precise algorithm is presented in Algorithm 2. Now, we will analyze its time complexity. Assume $(G, A, \tilde{R})$ is a fuzzy formal context, and the cardinality of objects and attributes are $|G|$ and $|A|$. The cardinality of objects in the information granule $\left(X,(B,C)\right)$ is $|X|$ and the cardinality of positive attributes and negative attributes are $|B|$ and $|C|$. Then extent and intent of information granule $\left(X,(B,C)\right)$ are Y^F and $(B,C)^H$, respectively. In the Step 1, the time complexity is $O(|G_1||A_1|)(|Y^F_1| + |(B_1,C_1)^H|)$ based on Algorithm 1. Running Steps 3-16 take $O(|G_i||A_i|)(|X^I_i||Y^F_i||(B,C)^H_i|)$. Hence, running Steps 2-19 take $O(n|G_n||A_n|)(|X^I_n||Y^F_n||(B,C)^H_n|)$ where n is the number of cognitive state. Therefore, the time complexity of Algorithm 2 is $O(n|G_n||A_n|)(|X^I_n||Y^F_n||(B,C)^H_n|)$. Then an example describes the calculation process.

Example 6. In Example 1, we know that five patients have six symptoms. As time progresses, there will be more patients (such as patients 6, 7, 8, 9 and 10) from whom other symptoms, such as abdominal distention, jaundice, anxiety and nausea will be observed. The information is depicted in Table 2. It should be noticed that the fuzzy membership degrees of patients 1, 2, 3, 4 and 5 under the new symptoms abdominal distention, jaundice, anxiety and nausea do not have to be "0" owing to new symptoms that can be tested in the original five patients in the later treatment.

Five original patients and six original symptoms in Table 1 are denoted by $G_1 = \{1,2,3,4,5\}$ and $A_1 = \{a,b,c,d,e,f\}$. Then the information granules are given by $X_1 = \{12\}$ and $(B,C)_1 = (be,a)$. Assume $\alpha = 0.6$ and $\gamma = 0.4$, then we have $F_1(X_1) = (bf,ad)$ and $H_1\left((B,C)_1\right) = \{24\}$.

Algorithm 2: Updating mechanism of VPO3Cs when objects in the information granule are added (UMVPO).

Input: A fuzzy formal context $(G, A, \tilde{R})$, α, γ, and given three-way information granules X_1 and $(B,C)_1$.
Output: VPO3Cs $(\mathcal{G}_1 \cap \mathcal{G}_2)^n$.

Initialize: $(\mathcal{G}_1 \cap \mathcal{G}_2)^1 = \{(Y^{\lessdot_{\alpha\gamma} \gtrdot_{\alpha\gamma}}, Y^{\lessdot_{\alpha\gamma}}) | Y \in Y_1^F\} \cup \{\big((E,F)^{\gtrdot_{\alpha\gamma}}, (E,F)^{\gtrdot_{\alpha\gamma} \lessdot_{\alpha\gamma}}\big) | (E,F) \in (B,C)_1^H\}$ and $i = 2$.
while $i \leq n$ **do**
  Compute derived information granules X_i^I and $(B,C)_i^I$;
  Update the extent and intent of three-way information granules Y_i^F and $(B,C)_i^H$;
  for each $X \in X_i^I$ **do**
    for each $Y \in Y_i^F$ **do**
      Compute the VPO3C $\big(H_i F_i(Y), F_i(Y)\big)$ from Eq. (4);
    end for
    for each $(E,F) \in (B,C)_i^H$ **do**
      if $\triangle(B,C)_{i-1}^H = \emptyset$ **then**
        Compute the VPO3C $\Big(H_i\big((E,F)\big), F_i H_i\big((E,F)\big)\Big)$ from Eq. (5) or (8);
      else
        Compute the VPO3C $\Big(H_i\big((E,F)\big), F_i H_i\big((E,F)\big)\Big)$ from Eq. (6), (7) or (9);
      end if
    end for
  end for
  Set $(\mathcal{G}_1 \cap \mathcal{G}_2)^i \leftarrow \big(H_i F_i(Y), F_i(Y)\big)$ and $(\mathcal{G}_1 \cap \mathcal{G}_2)^i \leftarrow \Big(H_i\big((E,F)\big), F_i H_i\big((E,F)\big)\Big)$.
  $i \rightarrow i + 1$;
end while
Return $(\mathcal{G}_1 \cap \mathcal{G}_2)^n$.

Table 2
A fuzzy formal context.

G	a	b	c	d	e	f	g	h	i	j
1	0.35	1.00	0.10	0.35	0.90	0.50	0.65	0.70	0.20	0.30
2	0.80	0.45	1.00	0.20	0.50	0.65	0.45	0.25	0.45	0.00
3	0.25	0.80	0.50	0.25	0.85	0.80	0.70	0.55	0.30	0.40
4	0.35	0.60	0.20	0.30	0.35	0.65	0.65	0.30	0.21	0.35
5	0.30	0.75	0.85	0.20	0.80	0.70	0.80	0.69	0.15	0.47
6	0.47	0.25	0.56	0.30	0.00	0.84	0.90	0.80	0.30	0.78
7	0.86	0.85	0.67	0.20	0.70	0.65	0.00	0.30	0.35	0.49
8	0.21	0.48	0.49	0.40	0.60	0.78	0.70	0.50	0.28	0.57
9	0.51	0.89	0.53	0.35	0.30	0.70	0.68	0.60	0.37	0.00
10	0.34	0.65	0.65	0.49	0.75	0.65	0.85	0.90	0.29	0.15

Furthermore, we denote the new patients with heart disease by 6,7,8,9,10 and new symptoms by g, h, i, j, respectively. Let $G_2 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A_2 = \{a, b, c, d, e, f, g, h, i, j\}$, $X_2 = \{1, 2, 7, 8\}$, $(B,C)_2 = (be, a)$, $\triangle G_1 = G_2 - G_1 = \{6, 7, 8, 9, 10\}$, $\triangle A_1 = \{g, h, i, j\}$, $\triangle X_1 = \{7, 8\}$, $\triangle(B,C)_1 = \emptyset$. First, assuming $X^I = X_2$ and $(B,C)^I = (B,C)_2$,

1. If $Y^F = X^I = \{1278\}$, then $Y_1^F = \{12\}$ and $\triangle Y_1^F = \{78\}$,
2. If $Y^F = X^I \cup H_2\big((B,C)^I\big) = \{1246789\}$, then $Y_1^F = \{124\}$ and $\triangle Y_1^F = \{6789\}$,
3. If $Y^F = X^I \cap H_2\big((B,C)^I\big) = \{278\}$, then $Y_1^F = \{2\}$ and $\triangle Y_1^F = \{78\}$,
4. If $(B,C)^H = (B,C)_2 = (be, a)$, then $(B,C)_1^H = (be, a)$ and $\triangle(B,C)_1^H = \emptyset$,
5. If $(B,C)^H = F_2(X^I) \cap (B,C)^I = \emptyset$, then $(B,C)_1^H = \emptyset$ and $\triangle(B,C)_1^H = \emptyset$,
6. If $(B,C)^H = F_2(X^I) \cup (B,C)^I = (befg, ai)$, then $(B,C)_1^H = (bef, a)$ and $\triangle(B,C)_1 = (g, i)$.

Therefore, there are four non-empty VPO3Cs which are $\big(127, (fg, i)\big)$, $\big(1246789, (befg, ai)\big)$, $\big(246789, (beg, ai)\big)$ and $\big(27, (f, i)\big)$.

Secondly, assuming $(B,C)^I = (B,C)_i$ and $X^I = X_2 \cap H_2\big((B,C)^I\big)$,

1. If $(B,C)^H = F_2(X^I) \cup (B,C)^I = (beg, ai)$, then $(B,C)_1^H = (be, a)$ and $\triangle(B,C)_1^H = (g, i)$. Thus, VPO3C is $\big(246789, (beg, ai)\big)$.

Finally, suppose $X^I = X_2 \cup H_2\big((B,C)^I\big)$ and $(B,C)^I = (B,C)_i$,

1. If $(B,C)^H = F_2(X^I) \cap (B,C)^I = (be, a)$, then $(B,C)_1^H = (be, a)$ and $\triangle(B,C)_1^H = \emptyset$,

2. If $(B,C)^H = F_2(X^I) \cup (B,C)^I = (befg, ai)$, then $(B,C)_1^H = (bef, a)$ and $\triangle(B,C)_1 = (g, i)$. Hence, VPO3Cs are $\big(246789, (beg, ai)\big)$ and $\big(1246789, (befg, ai)\big)$.

To sum up, we know that there exist four different VPO3Cs from the state F_1, H_1 to F_2, H_2.

5.3. Updating mechanism when the attributes in the information granules are added

Similar to subsection 5.2, we thoroughly elaborate the updating mechanism when the attributes in the information granule are added in this subsection. In fact, three derived information granules $(B,C)_i, (B,C)_i \cap X_i^{\lessdot_{\alpha\gamma}}, (B,C)_i \cup X_i^{\lessdot_{\alpha\gamma}}$ will vary under the premise of changed information granule $(B,C)_i$, where i represents the i-th state. To acquire more knowledge, we develop methods to dynamically update the VPO3Cs.

Proposition 4. *Let $(G,A,\tilde{R})$ be a fuzzy formal context with $0 \le \gamma < \alpha \le 1$, $\triangle X_{i-1} = \emptyset$ and $\triangle(B,C)_{i-1} \neq \emptyset$. Assuming $X^I = X_i$ and $(B,C)^I = (B,C)_i$, then the following statements hold:*

1. If $(B,C)^H = (B,C)^I$, then $\triangle(B,C)^H_{i-1} = (B,C)_i - (B,C)_{i-1}$,

2. If $(B,C)^H = F_i(X^I) \cup (B,C)^I$, then command $(B,C)^H_{i-1} = F_{i-1}(X_{i-1}) \cap F_{\triangle X_{i-1}}(\triangle X_{i-1}) \cup (B,C)_{i-1}$ and $\triangle(B,C)^H_{i-1} = F_{\triangle(B,C)_{i-1}}(X_i) \cup \triangle(B,C)_{i-1}$,

3. If $(B,C)^H = F_i(X^I) \cap (B,C)^I$, then command $(B,C)^H_{i-1} = F_{i-1}(X_{i-1}) \cap F_{\triangle X_{i-1}}(\triangle X_{i-1}) \cap (B,C)_{i-1}$ and $\triangle(B,C)^H_{i-1} = F_{\triangle(B,C)_{i-1}}(X_i) \cap \triangle(B,C)_{i-1}$.

Then VPO3Cs are updated as follows:

$$\begin{aligned}
&\Big(H_i\big((B,C)^H\big), F_iH_i\big((B,C)^H\big)\Big) = \Bigg(H_{i-1}\big((B,C)^H_{i-1}\big) \cup H_{\triangle X_{i-1}}\big((B,C)^H_{i-1}\big) \cup H_{\triangle(B,C)^H_{i-1}}\big(\triangle(B,C)^H_{i-1}\big),\\
&F_{i-1}\Big(H_{i-1}\big((B,C)^H_{i-1}\big) \cup \Big(H_{\triangle(B,C)^H_{i-1}}\big(\triangle(B,C)^H_{i-1}\big) \cap G_{i-1}\Big)\Big)\\
&\cap F_{\triangle X_{i-1}}\Big(H_{\triangle X_{i-1}}\big((B,C)^H_{i-1}\big) \cup \Big(H_{\triangle(B,C)^H_{i-1}}\big(\triangle(B,C)^H_{i-1}\big) \cap \triangle G_{i-1}\Big)\Big)\\
&\cup F_{\triangle(B,C)^H_{i-1}}\Big(H_{i-1}\big((B,C)^H_{i-1}\big) \cup H_{\triangle X_{i-1}}\big((B,C)^H_{i-1}\big) \cup H_{\triangle(B,C)^H_{i-1}}\big(\triangle(B,C)^H_{i-1}\big)\Big)\Bigg).
\end{aligned} \tag{10}$$

4. If $Y^F = X^I$, then $\triangle Y^F_{i-1} = X_i - X_{i-1} = \emptyset$,

5. If $Y^F = X^I \cap H_i\big((B,C)^I\big)$, then command $Y^F_{i-1} = X_{i-1} \cap \Big(H_{i-1}\big((B,C)_{i-1}\big) \cup H_{\triangle(B,C)_{i-1}}\big(\triangle(B,C)_{i-1}\big)\Big)$ and $\triangle Y^F_{i-1} = X_{i-1} \cap H_{\triangle X_{i-1}}\big((B,C)_{i-1}\big) = \emptyset$,

6. If $Y^F = X^I \cup H_i\big((B,C)^I\big)$, then command $Y^F_{i-1} = X_{i-1} \cup H_{i-1}\big((B,C)_{i-1}\big) \cup \Big(H_{\triangle(B,C)_{i-1}}\big(\triangle(B,C)_{i-1}\big) \cap G_{i-1}\Big)$ and $\triangle Y^F_{i-1} = \Big(H_{\triangle(B,C)_{i-1}}\big(\triangle(B,C)_{i-1}\big) \cap \triangle G_{i-1}\Big) \cup H_{\triangle X_{i-1}}\big((B,C)_{i-1}\big)$.

Therefore, VPO3Cs are updated as follows:

$$\begin{aligned}
&\big(H_iF_i(Y^F), F_i(Y^F)\big) = \Big(H_i\big(F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1}) \cup F_{\triangle(B,C)_{i-1}}(Y^F)\big), F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1}) \cup F_{\triangle(B,C)_{i-1}}(Y^F)\Big)\\
&= \Big(H_{i-1}\big(F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1})\big) \cup H_{\triangle Y^F_{i-1}}\big(F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1})\big) \cup H_{\triangle(B,C)_{i-1}}\big(F_{\triangle(B,C)_{i-1}}(Y^F)\big),\\
&F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1}) \cup F_{\triangle(B,C)_{i-1}}(Y^F)\Big).
\end{aligned} \tag{11}$$

Proof. This proof is similar to Proposition 1. □

Proposition 4 indicates that there exist six VPO3Cs when $X^I = X_i$ and $(B,C)^I = (B,C)_i$. However, when $(B,C)^I = F_i(X^I) \cap (B,C)_i$ and $(B,C)^I = F_i(X^I) \cup (B,C)_i$, there will be some duplicate three-way concepts after the same updating process, and here we only focus on different three-way concepts.

Proposition 5. *Let $(G,A,\tilde{R})$ be a fuzzy formal context with $0 \le \gamma < \alpha \le 1$, $\triangle X_{i-1} = \emptyset$ and $\triangle(B,C)_{i-1} \neq \emptyset$. Assuming $X^I = X_i$ and $(B,C)^I = F_i(X^I) \cap (B,C)_i$, then we have $(B,C)^I_{i-1} = F_{i-1}(X_{i-1}) \cap F_{\triangle X_{i-1}}(\triangle X_{i-1}) \cap (B,C)_{i-1}$ and $\triangle(B,C)^I_{i-1} = F_{\triangle(B,C)_{i-1}}(X_i) \cap \triangle(B,C)_{i-1}$. The following statements hold:*

1. If $Y^F = X^I \cap H_i\big((B,C)^I\big)$, then command $Y^F_{i-1} = X_{i-1} \cap \Big(H_{i-1}\big((B,C)^I_{i-1}\big) \cup H_{\triangle(B,C)^I_{i-1}}\big(\triangle(B,C)^I_{i-1}\big)\Big)$ and $\triangle Y^F_{i-1} = X_{i-1} \cap H_{\triangle X_{i-1}}\big((B,C)^I_{i-1}\big) = \emptyset$,

2. If $Y^F = X^I \cup H_i\big((B,C)^I\big)$, then command $Y^F_{i-1} = X_{i-1} \cup H_{i-1}\big((B,C)^I_{i-1}\big) \cup \Big(H_{\triangle(B,C)^I_{i-1}}\big(\triangle(B,C)^I_{i-1}\big) \cap G_{i-1}\Big)$ and $\triangle Y^F_{i-1} = \Big(H_{\triangle(B,C)^I_{i-1}}\big(\triangle(B,C)^I_{i-1}\big) \cap \triangle G_{i-1}\Big) \cup H_{\triangle X_{i-1}}\big((B,C)^I_{i-1}\big)$.

Therefore, VPO3Cs are updated as follows:

$$\begin{aligned}
&\big(H_iF_i(Y^F), F_i(Y^F)\big) = \Big(H_i\big(F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1}) \cup F_{\triangle(B,C)_{i-1}}(Y^F)\big), F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1}) \cup F_{\triangle(B,C)_{i-1}}(Y^F)\Big)\\
&= \Big(H_{i-1}\big(F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1})\big) \cup H_{\triangle Y^F_{i-1}}\big(F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1})\big) \cup H_{\triangle(B,C)_{i-1}}\big(F_{\triangle(B,C)_{i-1}}(Y^F)\big),\\
&F_{i-1}(Y^F_{i-1}) \cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1}) \cup F_{\triangle(B,C)_{i-1}}(Y^F)\Big).
\end{aligned} \tag{12}$$

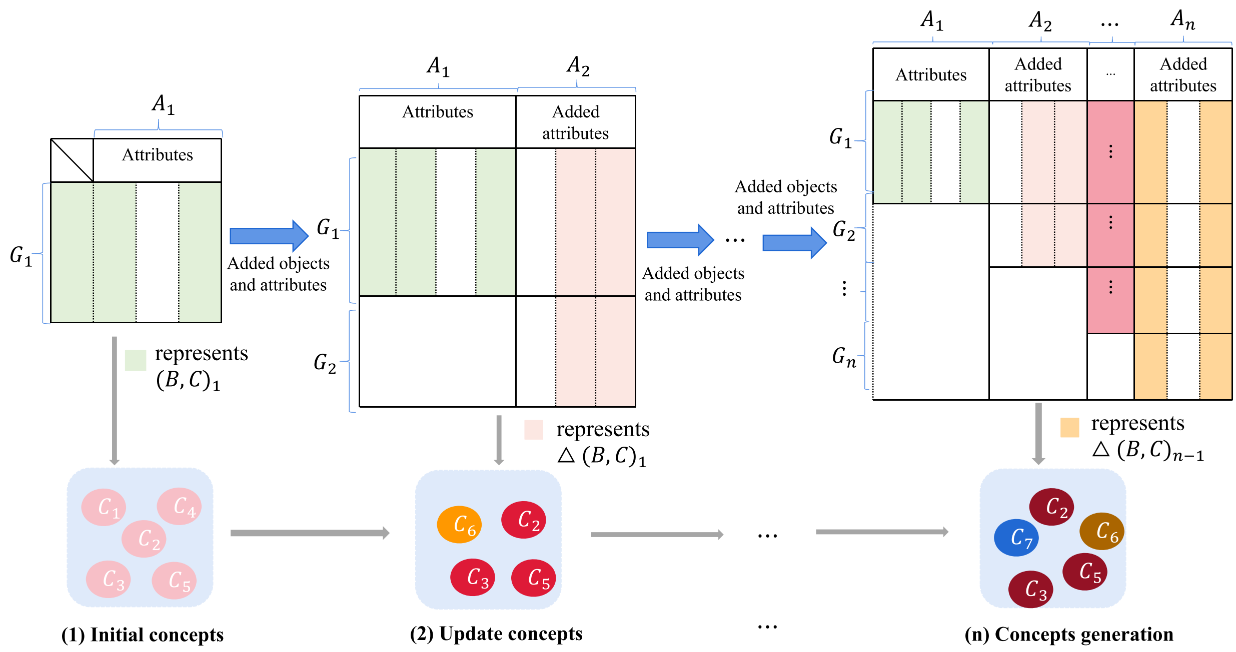

Fig. 5. The dynamic learning process of $\alpha-\gamma$ three-way concepts when the attributes in the information granules are added.

Proposition 6. *Let $(G,A,\tilde{R})$ be a fuzzy formal context with $0\le\gamma<\alpha\le 1$, $\triangle X_{i-1}=\emptyset$ and $\triangle(B,C)_{i-1}\neq\emptyset$. Assuming $X^I=X_i$ and $(B,C)^I=F_i(X^I)\cup(B,C)_i$, then we have $(B,C)^I_{i-1}=F_{i-1}(X_{i-1})\cap F_{\triangle X_{i-1}}(\triangle X_{i-1})\cup(B,C)_{i-1}$ and $\triangle(B,C)^I_{i-1}=F_{\triangle(B,C)_{i-1}}(X_i)\cup\triangle(B,C)_{i-1}$. The following statement holds:*

1. *If $Y^F=X^I\cap H_i\big((B,C)^I\big)$, then command $Y^F_{i-1}=X_{i-1}\cap\Big(H_{i-1}\big((B,C)^I_{i-1}\big)\cup H_{\triangle(B,C)^I_{i-1}}\big(\triangle(B,C)^I_{i-1}\big)\Big)$ and $\triangle Y^F_{i-1}=\emptyset$. Therefore, VPO3C is updated as follows:*

$$
\begin{aligned}
&\big(H_iF_i(Y^F),F_i(Y^F)\big)=\Big(H_i\big(F_{i-1}(Y^F_{i-1})\cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1})\cup F_{\triangle(B,C)_{i-1}}(Y^F)\big),F_{i-1}(Y^F_{i-1})\cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1})\cup F_{\triangle(B,C)_{i-1}}(Y^F)\Big)\\
&=\Big(H_{i-1}\big(F_{i-1}(Y^F_{i-1})\cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1})\big)\cup H_{\triangle Y^F_{i-1}}\big(F_{i-1}(Y^F_{i-1})\cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1})\big)\cup H_{\triangle(B,C)_{i-1}}\big(F_{\triangle(B,C)_{i-1}}(Y^F)\big),\\
&F_{i-1}(Y^F_{i-1})\cap F_{\triangle Y^F_{i-1}}(\triangle Y^F_{i-1})\cup F_{\triangle(B,C)_{i-1}}(Y^F)\Big).
\end{aligned}
\tag{13}
$$

According to Propositions 4-6, three-way updating concepts after adding attributes in the information granule can be obtained. We can see that a total of nine three-way updating concepts can be generated from the current information state F_{i-1},H_{i-1} to F_i,H_i with $\triangle X_{i-1}=\emptyset$ and $\triangle(B,C)_{i-1}\neq\emptyset$. Practically, the resulting concept might be less than or equal to nine. Fig. 5 describes the dynamic concept learning process by recursive approach based on VPO3Cs.

In addition, Algorithm 3 is an updating algorithm for the adding attributes in the information granule. Running Step 1 takes $O(|G_1||A_1|)(|Y^F_1|+|(B_1,C_1)^H|)$ based on Algorithm 1. In the Steps 3-12, the time complexity is $O(|G_i||A_i|)(|(B,C)^I_i||(B,C)^H_i||Y^F_i|)$. Hence, running Steps 2-15 take $O(n|G_n||A_n|)(|(B,C)^I_n||(B,C)^H_n||Y^F_n|)$ where n is the number of cognitive state. Therefore, the time complexity of Algorithm 3 is $O(n|G_n||A_n|)(|(B,C)^I_n||(B,C)^H_n||Y^F_n|)$.

Example 7. Continued with Example 6, suppose that $X_2=\{12\}$ and $(B,C)_2=(begh,aij)$, then we obtain $\triangle X_1=\emptyset$ and $\triangle(B,C)_1=(gh,ij)$. First, assuming $X^I=X_2$ and $(B,C)^I=(B,C)_2$, then

1. If $(B,C)^H=(B,C)^I=(begh,aij)$, then $(B,C)^H_1=(be,a)$ and $\triangle(B,C)^H_1=(gh,ij)$,
2. If $(B,C)^H=F_2(X^I)\cup(B,C)^I=(befgh,aij)$, then $(B,C)^H_1=(bef,a)$ and $\triangle(B,C)_1=(gh,ij)$,
3. If $(B,C)^H=F_2(X^I)\cap(B,C)^I=(\emptyset,i)$, then $(B,C)^H_1=(\emptyset,\emptyset)$ and $\triangle(B,C)^H_1=(\emptyset,i)$,
4. If $Y^F=X^I=\{12\}$, then $Y^F_1=\{12\}$ and $\triangle Y^F_1=\emptyset$,
5. If $Y^F=X^I\cap H_2\big((B,C)^I\big)=\{2\}$, then $Y^F_1=\{2\}$ and $\triangle Y^F_1=\emptyset$,
6. If $Y^F=X^I\cup H_2\big((B,C)^I\big)=\{123456789\}$, then $Y^F_1=\{12345\}$ and $\triangle Y^F_1=\{6789\}$,

Therefore, there are four different non-empty VPO3Cs which are $\big(123456789,(bcefgh,aij)\big)$, $\big(2,(\emptyset,i)\big)$, $\big(12,(f,i)\big)$ and $\big(23456789,(begh,aij)\big)$.

Next, suppose $X^I=X_2$ and $(B,C)^I=F_i(X^I)\cap(B,C)_i$,

1. $Y^F=X^I\cap H_2\big((B,C)^I\big)=\{2\}$, then $Y^F_1=\{2\}$ and $\triangle Y^F_1=\emptyset$,

Algorithm 3: Updating mechanism of VPO3Cs when attributes in the information granules are added (UMVPA).

Input: A fuzzy formal context $(G, A, \tilde{R})$, α, γ, and given three-way information granules X_1 and $(B,C)_1$.
Output: VPO3Cs $(\mathcal{G}_1 \cap \mathcal{G}_2)^n$.

Initialize: $(\mathcal{G}_1 \cap \mathcal{G}_2)^1 = \{(Y^{\lessdot_{\alpha\gamma} \gtrdot_{\alpha\gamma}}, Y^{\lessdot_{\alpha\gamma}}) | Y \in Y_1^F\} \cup \{\big((E,F)^{\gtrdot_{\alpha\gamma}}, (E,F)^{\gtrdot_{\alpha\gamma} \lessdot_{\alpha\gamma}}\big) | (E,F) \in (B,C)_1^H\}$ and $i = 2$.
while $i \le n$ **do**
  Calculate information granules X_i^I and $(B,C)_i^I$;
  Update the extent and intent of three-way information granules $(B,C)_i^H$ and Y_i^F;
  for each $(B,C) \in (B,C)_i^I$ **do**
    for each $(E,F) \in (B,C)_i^H$ **do**
      Compute the VPO3C $\Big(H_i\big((E,F)\big), F_i H_i\big((E,F)\big)\Big)$ from Eq. (10);
    end for
    for each $Y \in Y_i^F$ **do**
      Compute the VPO3C $\big(H_i F_i(Y), F_i(Y)\big)$ from Eq. (11), (12) or (13);
    end for
  end for
  Set $(\mathcal{G}_1 \cap \mathcal{G}_2)^i \leftarrow \Big(H_i\big((E,F)\big), F_i H_i\big((E,F)\big)\Big)$ and $(\mathcal{G}_1 \cap \mathcal{G}_2)^i \leftarrow \big(H_i F_i(Y), F_i(Y)\big)$.
  $i \rightarrow i+1$;
end while
Return $(\mathcal{G}_1 \cap \mathcal{G}_2)^n$.

2. $Y^F = X^I \cup H_2\big((B,C)^I\big) = \{12\}$, then $Y_1^F = \{12\}$ and $\triangle Y_1^F = \emptyset$. Thus, non-empty VPO3Cs are $\big(2, (\emptyset, i)\big)$ and $\big(12, (f, i)\big)$.

At last, assuming $X^I = X_2$ and $(B,C)^I = F_i(X^I) \cup (B,C)_i$,

1. $Y^F = X^I \cap H_2\big((B,C)^I\big) = \{12\}$, then $Y_1^F = \{12\}$ and $\triangle Y_1^F = \emptyset$. Hence, non-empty VPO3C is $\big(12, (f, i)\big)$.

In summary, we conclude from Example 7 that there are four non-empty VPO3Cs from the state F_1, H_1 to F_2, H_2 with the adding attributes in the information granule.

6. Experimental evaluation

In this section, a series of comparative experiments are conducted. To demonstrate the effectiveness of these algorithms, we will verify them from the perspective of running time and the number of concepts. In fact, although VPO3C can provide a more comprehensive description of conceptual information through positive and negative information, this will lead to longer time-consuming and smaller numbers of concepts compared with two-way concepts. Therefore, interval-valued data [19] are selected to learn interval-value concepts and then compare with the proposed algorithms in this paper. We first compare LVPTC algorithm with the CCL approach in static concept learning, and then compare UMVPO and UMVPA algorithms with several CCL algorithms [19] from dynamic concept learning.

In order to illustrate the effectiveness of UMVPO and UMVPA algorithms, we divide the datasets into two groups as shown in Table 3, where the datasets 1-5 are used to verify the dynamic updating mechanism of algorithm UMVPO, and the datasets 6-10 are used to validate the dynamic updating mechanism of algorithm UMVPA. For the sake of fairness, the above experiments are performed using in MATLAB 2015b on a personal computer with Intel(R) Core(TM) i7-4790 CPU @ 3.6GHz and 16 GB memory.

In the experiment, ten public datasets are downloaded from UCI Machine learning Repository (see https://uci.edu/) which are shown in Table 3. In fact, most of the selected datasets are continuous, except for individual attributes that are discrete. For the non-continuous attributes, the dataset is fuzzified to represent the membership degree belonging to the interval $[0,1]$ by using $\frac{f(x,a)-a_{min}}{a_{max}-a_{min}}$ in [6], where $f(x,a)$ means the value of object x under attribute a, and a_{max} and a_{min} are the maximum and minimum values of all objects with attribute a, respectively. Besides, the lower limit and upper limit of the interval-valued attributes as mentioned in [19] are constructed as follows:

1. $a^L(x) = max\big(0, (1-\varepsilon) \times f(x,a)\big)$;
2. $a^U(x) = min\big(1, (1+\varepsilon) \times f(x,a)\big)$.

where $a^L(x)$ and $a^U(x)$ represent the lower limit and upper limit of object x in attribute a, respectively. In addition, from Definition 6, it is obvious that the different values of parameters α and γ can influence different optimal performance of the proposed algorithms. Consequently, we set α, γ and ε to a value between 0.1 and 1.0 in steps of 0.1. The experimental results in the following tables are shown with the optimal parameters. For the fairness of the experiment, we conducted ten trials to obtain the average numbers of concepts on each dataset.

6.1. Compare the performance of concept learning

In this subsection, to verify the effectiveness of the algorithm LVPTC in static learning, we compare it with OIvA-IvCL in terms of numbers of concepts and running time, and then evaluate the ability to produce the knowledge by the two algorithms.

Firstly, to obtain the initial information granule, we randomly select 40%, 60%, 70%, 90% objects of each dataset as its initial object subset X_0, and select 40%, 60%, 70%, 90% attributes of each dataset as its initial attribute subset (B_0, C_0). Therefore, there exist 4×4 pairs about the number of the initial information granule in each dataset. The evaluation results are represented in Tables 4 and 5 and the best results are marked in bold.

Table 3
Data description.

No.	Data sets	Objects	Attributes	Classes
1	Led-display	1000	7	10
2	Turkiye	5820	32	3
3	Nursery	12960	8	3
4	Mushroom	8124	22	2
5	SpliceEW	5000	40	3
6	Pengleuk	72	7070	3
7	Pengnci9	60	9713	9
8	Gene8	96	7129	3
9	Lung-cancer	181	12533	2
10	Gene2	174	12533	11

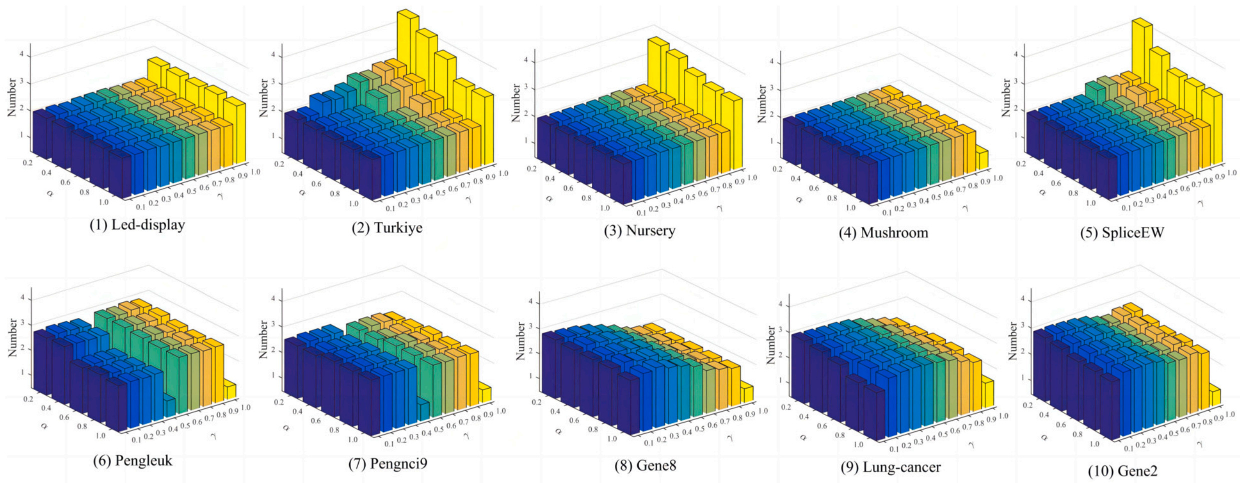

Fig. 6. Number of three-way concepts varying with α and γ.

The optimal α and the numbers of VPO3Cs are recorded in Table 4 on ten datasets. On all datasets, the number of VPO3Cs is no more than six in the process of learning knowledge. Furthermore, the numbers of LVPTC are almost more than or equal to that of OIvA-IvCL, expect for Mushroom, which reveals the proposed model can mine more rich knowledge. It should be pointed out that the numbers of concepts learned by algorithms OIvA-IvCL and LVPTC in datasets 6-10 are approximate. This may be due to the high-dimensional attributes of datasets. Additionally, the results in Table 5, which shows that LVPTC is almost better than OIvA-IvCL in the average numbers, which reveals the effectiveness of VPO3Cs. As listed as Table 6, it records the time consumption of LVPTC compared with OIvA-IvCL in static data. From this table, we demonstrate that the proposed algorithm LVPTC can save more time to guarantee the ability to learn concepts. Then the elapsed time increases with the increase of data size, especially in large datasets. Therefore, compared with OIvA-IvCL, although the VPO3C introduces negative information to describe objects, the number of three-way concepts can be generated by adjusting the thresholds α and γ, which can acquire more knowledge to some extent.

In addition, Fig. 6 depicts the number curve of VPO3C varying with parameters α and γ, where they change from 0.1 to 1.0 with steps of 0.1. We can clearly see that the number of three-way concepts is relatively stable for most cases. Furthermore, most datasets present stability in learning three-way concepts in their respective regions. Therefore, we can select the optimal parameter based on the maximum number of three-way concepts from Fig. 6.

6.2. Compare the performance of dynamic concept learning

Dynamic concept learning aims to assess the performance of model to learn knowledge from new data. We have provided a detailed theoretical description for dynamic updating in Section 5. Next, we will conduct a performance analysis of dynamic concept learning ability on the certain selected datasets. Firstly, with respect to information granules, we randomly select 40% objects and 30% attributes of each dataset as its object subset X_0 and initial attribute subset (B_0, C_0), respectively. To generate dynamic dataset, each dataset is divided into initial set and incremental set. Thus, we choose the first 50% of dataset object as the initial object G_0 and the first 30% of dataset attribute as the initial attribute A_0. During incremental process, we divide the remaining objects and attributes into ten Batches (i.e., Batch 1, Batch 2, ⋯, Batch 10), and each Batch accounts for 5% of remaining objects and attributes. At last, we will implement ten incremental learning processes.

Table 7 records the numbers of concept for dynamic learning and the last column means the average numbers of concept. Next the optimal parameters ε, α and γ are listed in Table 8. From these results, we know that LVPTC, UMVPO and UMVPA are the same

Table 4
Number of three-way concepts with different algorithms.

No.s	Attributes / Objects	α, γ	40%		60%		70%		90%	
			OIvA-IvCL	LVPTC	OIvA-IvCL	LVPTC	OIvA-IvCL	LVPTC	OIvA-IvCL	LVPTC
1	40%		2.5	***3.3***	2.3	***3.0***	2.3	***2.8***	2.2	***2.5***
	60%	$\alpha = 0.4$	2.7	***3.4***	2.5	***3.0***	2.4	***2.7***	2.2	***2.3***
	70%	$\gamma = 1.0$	2.8	***3.4***	2.5	***2.9***	2.5	***2.7***	2.3	***2.3***
	90%		2.9	***3.3***	2.7	***3.0***	2.6	***2.7***	2.5	***2.6***
2	40%		3.0	***4.6***	3.0	***4.5***	3.0	***4.6***	3.0	***3.9***
	60%	$\alpha = 0.1$	3.0	***4.8***	3.0	***4.7***	3.0	***4.6***	3.0	***3.9***
	70%	$\gamma = 1.0$	3.0	***4.7***	3.0	***4.8***	3.0	***4.9***	3.0	***3.9***
	90%		3.0	***4.9***	3.0	***4.9***	3.0	***4.9***	3.0	***4.2***
3	40%		3.0	***3.9***	3.0	***3.7***	3.0	***3.8***	3.0	***3.4***
	60%	$\alpha = 0.4$	3.0	***4.0***	3.0	***3.8***	3.0	***3.5***	3.0	***3.2***
	70%	$\gamma = 1.0$	3.0	***4.0***	3.0	***3.7***	3.0	***3.7***	3.0	***3.5***
	90%		3.0	***4.0***	3.0	***3.8***	3.0	***3.8***	3.0	***3.3***
4	40%		***3.0***	2.0	***2.9***	2.0	***2.9***	2.0	***2.8***	2.0
	60%	$\alpha = 0.6$	***2.9***	2.1	***2.9***	2.0	***2.8***	2.0	***2.7***	2.0
	70%	$\gamma = 0.4$	***2.9***	2.1	***2.8***	2.0	***2.8***	2.1	***2.8***	2.0
	90%		***3.0***	2.1	***2.9***	2.2	***2.8***	2.1	***2.8***	2.0
5	40%		4.3	***4.6***	***4.2***	4.0	4.1	***4.2***	4.0	***4.2***
	60%	$\alpha = 0.2$	3.9	***4.3***	3.8	***4.1***	3.9	***4.0***	3.7	***4.1***
	70%	$\gamma = 1.0$	3.7	***4.8***	3.6	***4.2***	3.6	***4.3***	3.5	***4.2***
	90%		3.5	***4.5***	3.5	***4.4***	3.4	***4.5***	3.4	***4.2***
6	40%		***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	2.8
	60%	$\alpha = 0.3$	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	2.9
	70%	$\gamma = 0.6$	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	2.8
	90%		2.8	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	2.9
7	40%		***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	2.6
	60%	$\alpha = 0.7$	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	2.8	***3.0***	2.5
	70%	$\gamma = 0.3$	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	2.9	***3.0***	2.4
	90%		2.9	***3.0***	***3.0***	***3.0***	***3.0***	2.8	***3.0***	2.3
8	40%		***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	2.8
	60%	$\alpha = 0.3$	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	2.8
	70%	$\gamma = 0.1$	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***
	90%		***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***
9	40%		***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	2.8
	60%	$\alpha = 0.6$	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***
	70%	$\gamma = 0.3$	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	2.9
	90%		***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***
10	40%		***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	2.9
	60%	$\alpha = 0.7$	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***
	70%	$\gamma = 0.3$	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***
	90%		***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***

Table 5
Average number of three-way concepts.

Attributes / Objects	Alg	40%	60%	70%	90%
40%	OIvA-IvCL	3.08	3.04	3.03	***3.00***
	LVPTC	***3.34***	***3.22***	***3.24***	2.99
60%	OIvA-IvCL	3.05	3.02	3.01	2.96
	LVPTC	***3.36***	***3.26***	***3.16***	***2.97***
70%	OIvA-IvCL	3.04	2.99	2.99	2.96
	LVPTC	***3.40***	***3.26***	***3.26***	***3.00***
90%	OIvA-IvCL	3.01	3.01	2.98	2.97
	LVPTC	***3.38***	***3.33***	***3.28***	***3.05***

Table 6
Running time of learning three-way concepts with different algorithm (s).

No.s	Attributes / Objects	Alg	40%	60%	70%	90%	No.s	Attributes / Objects	Alg	40%	60%	70%	90%
1	40%	OIvA-IvCL	7.03	7.10	7.02	7.03	6	40%	OIvA-IvCL	1404.16	1432.79	1405.95	1403.98
		LVPTC	***3.96***	***4.37***	***4.94***	***4.82***			LVPTC	***183.23***	***208.00***	***215.71***	***237.49***
	60%	OIvA-IvCL	7.26	7.17	7.00	7.08		60%	OIvA-IvCL	1444.84	1401.25	1412.94	1405.16
		LVPTC	***3.96***	***4.48***	***4.61***	***5.17***			LVPTC	***184.51***	***205.43***	***210.25***	***236.82***
	70%	OIvA-IvCL	7.06	7.11	7.15	7.04		70%	OIvA-IvCL	1455.94	1444.99	1427.33	1413.72
		LVPTC	***4.11***	***4.56***	***4.66***	***4.82***			LVPTC	***182.01***	***200.69***	***208.87***	***238.99***
	90%	OIvA-IvCL	7.20	7.20	7.14	7.08		90%	OIvA-IvCL	1507.43	1420.28	1465.86	1413.77
		LVPTC	***4.30***	***4.65***	***4.70***	***4.94***			LVPTC	***179.74***	***196.37***	***208.82***	***238.47***
2	40%	OIvA-IvCL	58.20	60.46	57.58	***57.42***	7	40%	OIvA-IvCL	1747.02	1709.89	1676.31	1674.89
		LVPTC	***49.93***	***54.39***	***56.19***	70.78			LVPTC	***207.69***	***230.85***	***250.63***	***271.73***
	60%	OIvA-IvCL	57.91	58.49	57.41	***59.16***		60%	OIvA-IvCL	1695.01	1690.75	1676.99	1674.51
		LVPTC	***50.28***	***55.15***	***57.04***	71.56			LVPTC	***198.56***	***228.61***	***241.93***	***258.61***
	70%	OIvA-IvCL	57.29	58.90	57.47	***58.23***		70%	OIvA-IvCL	1693.01	1680.38	1679.32	1662.08
		LVPTC	***50.88***	***55.37***	***57.32***	71.96			LVPTC	***197.98***	***226.92***	***236.09***	***260.86***
	90%	OIvA-IvCL	58.97	58.04	58.34	***58.36***		90%	OIvA-IvCL	1650.96	1670.01	1678.76	1664.92
		LVPTC	***53.19***	***57.62***	***56.71***	74.20			LVPTC	***197.23***	***222.12***	***234.89***	***263.39***
3	40%	OIvA-IvCL	92.84	88.32	86.35	90.73	8	40%	OIvA-IvCL	1790.60	1791.26	1766.97	1762.07
		LVPTC	***31.77***	***33.84***	***35.01***	***36.13***			LVPTC	***234.00***	***260.32***	***272.82***	***304.27***
	60%	OIvA-IvCL	91.66	88.74	91.54	92.51		60%	OIvA-IvCL	1768.68	1759.22	1739.23	1757.05
		LVPTC	***33.23***	***35.51***	***36.38***	***37.81***			LVPTC	***228.24***	***252.14***	***269.13***	***298.47***
	70%	OIvA-IvCL	90.44	91.33	86.01	90.75		70%	OIvA-IvCL	1835.74	1782.26	1745.52	1742.42
		LVPTC	***34.55***	***36.75***	***37.98***	***39.23***			LVPTC	***224.47***	***251.04***	***268.91***	***297.69***
	90%	OIvA-IvCL	90.23	91.31	92.51	94.21		90%	OIvA-IvCL	1743.04	1736.62	1735.69	1755.41
		LVPTC	***38.13***	***40.48***	***41.67***	***42.41***			LVPTC	***232.89***	***266.48***	***280.81***	***310.23***
4	40%	OIvA-IvCL	91.34	88.53	90.33	91.04	9	40%	OIvA-IvCL	2722.73	2691.54	2697.47	2692.83
		LVPTC	***64.95***	***70.29***	***72.73***	***84.99***			LVPTC	***798.23***	***882.82***	***929.36***	***1040.02***
	60%	OIvA-IvCL	92.46	92.62	86.48	89.62		60%	OIvA-IvCL	2723.65	2689.36	2679.77	2670.63
		LVPTC	***66.95***	***70.29***	***72.73***	***84.99***			LVPTC	***782.46***	***860.63***	***915.38***	***1028.85***
	70%	OIvA-IvCL	91.61	91.01	90.68	88.79		70%	OIvA-IvCL	2663.37	2667.71	2674.37	2678.89
		LVPTC	***68.23***	***72.08***	***73.86***	***89.31***			LVPTC	***778.16***	***864.01***	***922.06***	***1032.59***
	90%	OIvA-IvCL	89.05	89.53	90.22	89.73		90%	OIvA-IvCL	2637.33	2656.31	2650.19	2647.74
		LVPTC	***72.31***	***75.69***	***77.78***	***89.67***			LVPTC	***778.37***	***880.16***	***931.94***	***1003.67***
5	40%	OIvA-IvCL	213.77	212.51	211.48	212.36	10	40%	OIvA-IvCL	2632.61	2644.82	2869.48	3047.96
		LVPTC	***43.00***	***46.56***	***48.38***	***60.24***			LVPTC	***761.02***	***856.89***	***903.85***	***989.84***
	60%	OIvA-IvCL	215.36	212.66	210.98	212.17		60%	OIvA-IvCL	2951.24	3008.67	3095.10	3101.52
		LVPTC	***42.85***	***46.69***	***48.46***	***60.27***			LVPTC	***751.80***	***844.13***	***889.54***	***962.78***
	70%	OIvA-IvCL	214.82	212.45	211.56	213.12		70%	OIvA-IvCL	3070.35	2868.61	3112.40	3133.11
		LVPTC	***42.53***	***46.54***	***48.14***	***59.91***			LVPTC	***727.64***	***817.83***	***880.35***	***990.97***
	90%	OIvA-IvCL	217.79	213.87	211.71	211.11		90%	OIvA-IvCL	3106.81	3109.82	3118.40	3110.12
		LVPTC	***43.76***	***47.96***	***49.23***	***61.12***			LVPTC	***750.32***	***850.65***	***891.00***	***976.86***

with the number of concept generation, which implies that concept learning in a dynamic environment is meaningful. Obviously, the numbers of concept of UMVPO and UMVPA are more than OIvA-IvCL on all datasets with the incremental process. Meanwhile, the last column in Table 7 shows that the average numbers of generating concepts of UMVPO and UMVPA are excellent on ten datasets, which verifies the updating concept learning proposed in this paper is effective.

Table 9 displays the running time of these algorithms for dynamic learning in optimal parameters, in which the time of algorithm taking less time is represented in bold. Obviously, UMVPO and UMVPA are the fastest on ten datasets compared with OIvA-IvCL and LVPTC. They can calculate the concept updating for dataset Lung-cancer containing 181 objects and 12533 attributes in approximatively 2 mins. As the objects and attributes increase, the running time increases in each dataset. It is evident that the differences of time consumption between static and dynamic concept learning are relatively small when the dimensionality of the dataset is small. Then the time gap increases with the data size increasing, which indicates that UMVPO and UMVPA have advantages in concept learning for big data.

In addition, further experiments are conducted to demonstrate running times varying with parameters α and γ. During the experiment, α and γ change from 0.1 to 1.0 with steps of 0.1. The running time of concept updating with α and γ is shown in Fig. 7. In fact, we can find that α and γ have a significant impact on the running time. Different values of α and γ imply that the positive and negative information is fused from a given fuzzy formal context, respectively. Most of data can effectively save running time in large areas. It is obvious that the time consumption is relatively long when γ is sufficiently large. In practice, we can select more approximate parameters based on the characteristics of data.

6.3. Experimental discussion and analysis

In the above subsection, we compare the performance of static concept learning and dynamic concept learning of two-way learning with other algorithms using ten data sets. Experimental results confirm the effectiveness and efficiency of our proposed algorithms

Table 7
Number of three-way concepts for dynamic learning in optimal parameters.

ID	Model	Batch 1	Batch 2	Batch 3	Batch 4	Batch 5	Batch 6	Batch 7	Batch 8	Batch 9	Batch 10	Average
1	OIvA-IvCL	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	3.0	3.0	2.20
	LVPTC	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.00
	UMVPO	***4.0***	***4.0***	***4.0***	***4.0***	***4.0***	***4.0***	***4.0***	***4.0***	***4.0***	***4.0***	***4.00***
2	OIvA-IvCL	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.00
	LVPTC	4.1	4.4	4.4	4.5	4.3	4.3	4.2	4.2	4.2	4.2	4.28
	UMVPO	***4.1***	***4.4***	***4.4***	***4.5***	***4.3***	***4.3***	***4.2***	***4.2***	***4.2***	***4.2***	***4.28***
3	OIvA-IvCL	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.00
	LVPTC	4.0	4.0	4.0	4.0	4.1	4.1	4.1	4.1	4.1	4.1	4.06
	UMVPO	***4.0***	***4.0***	***4.0***	***4.0***	***4.1***	***4.1***	***4.1***	***4.1***	***4.1***	***4.1***	***4.06***
4	OIvA-IvCL	2.4	2.3	2.0	2.0	3.4	3.0	2.6	2.6	2.0	2.0	2.43
	LVPTC	4.2	4.1	4.1	4.1	4.1	4.1	4.1	4.1	4.1	4.1	4.11
	UMVPO	***4.2***	***4.1***	***4.1***	***4.1***	***4.1***	***4.1***	***4.1***	***4.1***	***4.1***	***4.1***	***4.11***
5	OIvA-IvCL	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.00
	LVPTC	4.1	4.1	4.1	4.2	4.2	4.2	4.2	4.1	4.1	4.1	4.14
	UMVPO	***4.1***	***4.1***	***4.1***	***4.2***	***4.2***	***4.2***	***4.2***	***4.1***	***4.1***	***4.1***	***4.14***
6	OIvA-IvCL	2.6	2.3	2.2	2.2	2.2	2.2	2.2	2.2	2.1	2.1	2.23
	LVPTC	2.9	2.5	2.4	2.8	2.8	2.8	2.8	2.8	2.9	2.9	2.76
	UMVPA	***2.9***	***2.5***	***2.4***	***2.8***	***2.8***	***2.8***	***2.8***	***2.8***	***2.9***	***2.9***	***2.76***
7	OIvA-IvCL	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.00
	LVPTC	2.9	2.9	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0	2.98
	UMVPA	***2.9***	***2.9***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***2.98***
8	OIvA-IvCL	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.00
	LVPTC	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.3	3.3	3.06
	UMVPA	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.0***	***3.3***	***3.3***	***3.06***
9	OIvA-IvCL	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.00
	LVPTC	3.8	3.7	3.0	3.2	3.2	3.2	3.1	3.1	3.1	3.1	3.25
	UMVPA	***3.8***	***3.7***	***3.0***	***3.2***	***3.2***	***3.2***	***3.1***	***3.1***	***3.1***	***3.1***	***3.25***
10	OIvA-IvCL	2.7	2.7	2.7	2.7	2.7	2.7	2.7	2.7	2.7	2.7	2.70
	LVPTC	3.1	3.7	3.8	4.0	4.2	4.2	4.3	3.8	4.1	4.1	3.93
	UMVPA	***3.1***	***3.7***	***3.8***	***4.0***	***4.2***	***4.2***	***4.3***	***3.8***	***4.1***	***4.1***	***3.93***

Table 8
Optimal parameters of dynamic leaning.

No.	ε	α	γ
1	0.6	0.5	1.0
2	0.4	0.2	1.0
3	0.5	0.2	1.0
4	0.3	0.1	1.0
5	0.8	0.1	0.9
6	0.2	0.4	0.7
7	0.4	0.6	0.4
8	0.7	0.9	0.5
9	0.5	0.5	0.9
10	0.2	0.8	0.8

from numbers of concept and running time. Specifically, these advantages are reflected in as follows. 1) We could counterpoise the sensitivity and specificity of VPO3C to capture rich information by adjusting the threshold, thus facilitating the interpretability and understanding of cognitive process. 2) A simple method to learn unknown object from arbitrary information granule, which greatly reduce time consumption and reveal the systematic law of the human brain. 3) Dynamic mechanism of VPO3C is updated as time changes, which allows the integration of new input information into the current three-way concept to improve the efficiency of concept learning. In summary, a novel two-way concept-cognitive learning based on three-way decision is introduced to enhance cognition.

We have presented an interpretable two-way concept-cognitive model, but there exist some shortcomings. For instance, in the experiment, VPO3Cs might not necessarily have advantage compared with two-way concepts in terms of time because they need more time to acquire positive and negative information. Hence, it is necessary to optimize the operators proposed in this article.

Table 9
Running time for dynamic learning in optimal parameters (s).

ID	Model	Batch 1	Batch 2	Batch 3	Batch 4	Batch 5	Batch 6	Batch 7	Batch 8	Batch 9	Batch 10
1	OIvA-IvCL	4.83	5.21	5.60	6.00	6.42	6.87	7.37	8.89	8.42	8.98
	LVPTC	4.18	4.40	4.64	4.86	5.61	5.87	6.10	6.33	6.56	6.80
	UMVPO	***2.68***	***2.90***	***3.10***	***3.29***	***3.50***	***3.71***	***3.92***	***4.12***	***4.31***	***4.53***
2	OIvA-IvCL	40.56	43.34	46.38	49.72	53.36	57.26	61.25	65.80	70.59	75.59
	LVPTC	35.65	37.96	40.53	43.47	46.64	50.09	53.95	57.90	62.11	66.45
	UMVPO	***30.18***	***32.17***	***34.23***	***36.39***	***38.94***	***41.64***	***44.66***	***47.94***	***51.37***	***54.81***
3	OIvA-IvCL	77.20	82.80	88.85	95.30	102.23	109.47	117..51	125.85	134.84	143.89
	LVPTC	23.02	24.54	26.30	28.19	30.25	32.46	34.76	37.17	39.88	42.74
	UMVPO	***20.36***	***21.80***	***23.17***	***24.78***	***26.48***	***28.30***	***30.25***	***32.38***	***34.61***	***37.09***
4	OIvA-IvCL	298.56	313.57	330.80	351.34	376.02	405.23	438.39	477.57	523.52	573.51
	LVPTC	35.89	38.35	41.16	44.21	47.53	51.09	54.69	58.51	62.55	66.80
	UMVPO	***29.40***	***31.57***	***33.76***	***36.00***	***38.53***	***41.20***	***44.23***	***47.13***	***50.53***	***53.80***
5	OIvA-IvCL	86.67	89.87	94.02	99.45	106.83	115.33	125.85	137.49	151.62	167.39
	LVPTC	33.68	36.27	38.89	41.77	44.81	48.04	51.40	54.89	58.54	62.15
	UMVPO	***28.03***	***30.11***	***32.15***	***34.36***	***36.75***	***39.25***	***41.95***	***44.81***	***47.82***	***50.98***
6	OIvA-IvCL	555.86	580.16	612.15	652.74	699.35	754.04	816.38	888.49	975.99	1073.62
	LVPTC	35.40	37.60	40.05	42.73	45.65	48.92	52.60	56.78	61.44	66.71
	UMVPA	***31.48***	***33.36***	***35.58***	***38.05***	***40.74***	***43.69***	***46.81***	***50.42***	***54.37***	***58.66***
7	OIvA-IvCL	909.24	949.37	998.46	1058.77	1131.95	1235.82	1342.96	1467.15	1605.81	1759.16
	LVPTC	38.32	40.20	42.48	45.17	48.32	52.01	56.24	61.06	66.62	72.84
	UMVPA	***33.35***	***35.31***	***37.45***	***39.88***	***42.62***	***45.75***	***49.33***	***53.21***	***57.56***	***62.40***
8	OIvA-IvCL	769.22	796.82	838.55	889.42	952.13	1035.99	1127.33	1233.51	1354.61	1492.50
	LVPTC	36.57	38.51	40.70	43.28	46.27	49.74	53.80	58.37	63.41	69.05
	UMVPA	***31.69***	***33.62***	***35.76***	***38.02***	***40.64***	***43.61***	***46.95***	***50.66***	***54.86***	***59.53***
9	OIvA-IvCL	1353.58	1411.72	1487.09	1579.79	1690.72	1828.93	1990.81	2176.62	2382.41	2612.61
	LVPTC	118.32	123.91	130.81	139.17	149.07	160.69	173.89	189.09	206.24	225.73
	UMVPA	***94.04***	***98.99***	***104.84***	***111.60***	***119.58***	***128.56***	***138.80***	***150.31***	***163.24***	***177.63***
10	OIvA-IvCL	1253.45	1307.79	1378.09	1465.97	1566.03	1696.43	1845.50	2015.36	2203.07	2414.03
	LVPTC	106.14	111.36	117.81	125.48	134.46	144.82	156.50	169.69	184.67	201.70
	UMVPA	***84.52***	***89.08***	***94.53***	***100.88***	***108.14***	***116.27***	***125.39***	***135.55***	***147.01***	***159.64***

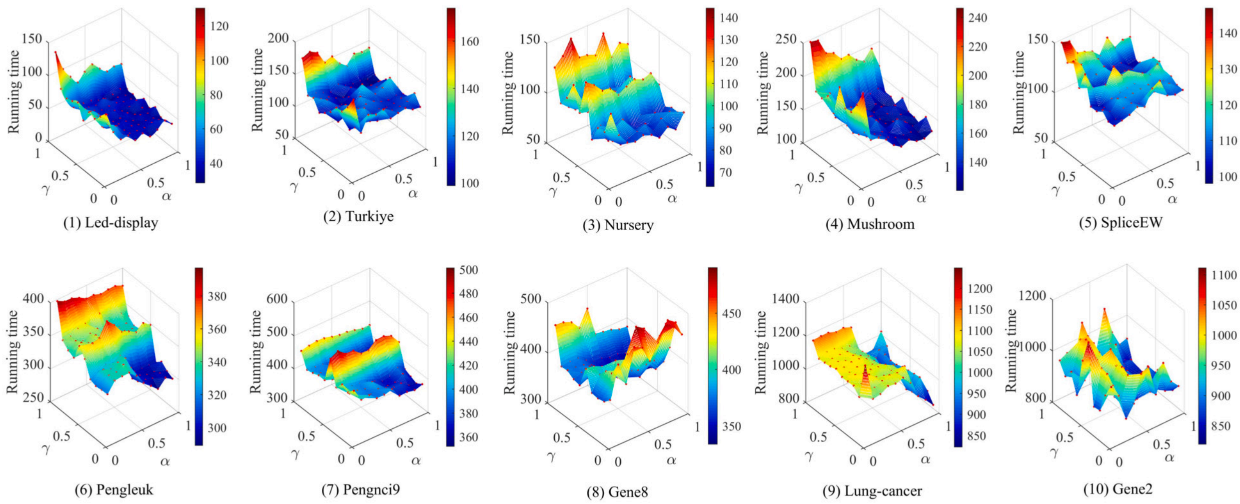

Fig. 7. Running time varying with α and γ.

7. Conclusion

With the rapid development of data, how to learn concepts from the given clue is a crucial issue, which can simulate the human brain cognitive process. In fact, some existing two-way learning methods ignore the flexibility of concepts, which make it difficult to choose the suitable concepts by setting thresholds. Thus, this article mainly discussed a novel TCCL based on three-way decision in a fuzzy formal context. Especially, the proposed variable precision object induced three-way concept settled the above problem and could achieve a great cognitive concept. The key of two-way learning is to learn more from the given clue through cognitive operators. Subsequently, our proposed two-way cognitive learning approach demonstrated the less time-consuming in the process of learning concepts from arbitrary information granules. As the updated concept was further learned, the cognitive mechanism integrated past experiences into itself for dynamic data and demonstrated that there exist nine methods of updating VPO3Cs from the current state F_{i-1}, H_{i-1} to F_i, H_i. As a result, the incremental learning mechanism can obtain diverse knowledge. Simultaneously, two dynamic updating algorithms UMVPO and UMVPA were proposed to update three-way concepts after adding objects and attributes. Finally, the proposed conclusions were demonstrated in comparative experiments on ten public datasets.

In summary, the current article proposes the dynamic updating mechanism of three-way model by considering the addition of objects and attributes in dynamic data. Nevertheless, there are still some limitations that need to be considered. For example, in comparison of two-way concepts, VPO3C might not necessarily have an advantage in terms of time consumption as they require more time to learn three-way information. Hence, to obtain a more efficient and interpretability theory of cognition, we need to consider how to optimize cognitive operators. Meanwhile, we mainly focus on continuous data values by setting thresholds, while ignoring multi-granularity formal concept analysis, especially in handing data with noise. Besides, two dynamic algorithms UMVPO and UMVPA do not consider how to update three-way concepts when objects and attributes are added simultaneously in the information granules. Based on the above discussion, our future work will continue to revolve around these themes.

CRediT authorship contribution statement

Chengling Zhang: Conceptualization, Methodology, Software, Writing – original draft. **Eric C.C. Tsang:** Formal analysis, Funding acquisition, Investigation, Supervision, Validation. **Weihua Xu:** Supervision, Validation. **Yidong Lin:** Software, Visualization. **Lanzhen Yang:** Data curation, Investigation. **Jiaming Wu:** Investigation, Software.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The authors are unable or have chosen not to specify which data has been used.

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