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Probabilistic linguistic three-way decisions: Integrating prospect theory with fuzzy possibilistic C-means clustering

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A B S T R A C T

This study proposes a novel multi-attribute decision-making (MADM) approach, Probabilistic Linguistic Three-Way Decision with Prospect Theory (PL-TWDP), which integrates Prospect Theory (PT) into the framework of Probabilistic Linguistic Term Sets (PLTSs). The method introduces an innovative technique for determining attribute weights by combining information entropy and deviation from the negative ideal solution (NIS), and develops a group satisfaction index to address limitations of traditional utility functions. The Fuzzy Possibilistic C-Means (FPCM) clustering algorithm is extended to refine the evaluation matrix in PLTSs, enhancing the identification of equivalent objects across clusters and the computation of conditional probabilities within the Three-Way Decision (TWD) model. By incorporating PT, the PL-TWDP method ranks alternatives based on utility perception values, promoting objective decision-making in probabilistic linguistic contexts. The effectiveness and superiority of the PL-TWDP method are demonstrated through a case study on air quality evaluation across multiple regions in China. Compared with eight established MADM methods, the proposed approach shows significant improvements in classification accuracy and robustness. Sensitivity analysis confirms its stability under varying parameter settings. This method effectively addresses uncertainties and risks in complex decision environments, providing a practical and reliable decision-making tool. Future work will focus on extending the method to handle large-scale group decision-making scenarios and further refining the integration of PT with PLTSs to enhance decision-making efficiency and accuracy.

1. Introduction

MADM is indeed an analytical process that focuses on identifying the most suitable solution from a finite collection of alternatives, each possessing multiple attributes, in order to attain a particular goal. Traditional MADM methods are primarily divided into two categories: decision-making approaches based on utility theory [1] and those grounded in dominance theory [2]. However, while these methods excel in handling MADM problems with clear and concrete evaluation values, they falter when faced with the fuzziness and categorical correspondence challenges of real-world scenarios, such as air quality evaluation across multiple regions.

In these complex contexts, decision-makers (DMs) not only seek the optimal solution but also pay attention to sets of samples with good or poor performance, hoping to extract valuable patterns through in-depth analysis of these samples. Unfortunately, traditional MADM methods cannot fulfill this requirement. If DMs could directly assess the quality of a particular period, such as a day, it

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would significantly enhance decision-making efficiency and reduce costs. However, due to the incompleteness of decision-making information, DMs often struggle to make accurate classifications, leading to increased classification errors.

While existing multi-attribute decision-making methods excel in handling clear and specific evaluation values, DMs often fall short when confronted with the ambiguity and classification challenges prevalent in real-world scenarios, such as cross-regional air quality assessments. Not only do traditional approaches struggle to manage uncertainty and risks within complex decision-making environments, but they also inadequately account for the psychological factors of decision-makers.

Notably, normal wiggly hesitant linguistic term sets possess the ability to delve deeper into uncertainties. DMs accommodate a broader spectrum of possible linguistic expressions and effectively capture the hesitancy inherent in their evaluations. As a result, they have the potential to mitigate classification errors to a certain degree. For instance, in the context of product quality evaluation, these term sets can more accurately represent the diverse levels of uncertainty in evaluators' minds. This is exemplified by the hesitation between descriptions such as "slightly good" and "moderately good", which they can articulate with greater precision.

To overcome this challenge, TWD have emerged. However, compared with normal wiggly hesitant linguistic term sets, TWD presents distinct and unparalleled advantages. TWD is far more than just a means of expressing uncertainty through linguistic terms and it is a comprehensive and systematic decision - making framework. When confronted with incomplete information, TWD has the ability to classify samples into positive, negative, and boundary regions, which is a crucial feature that differentiates it from other methods. To illustrate, let's consider the air quality evaluation scenario. In cases where weather information is incomplete, normal wiggly hesitant linguistic term sets merely express the existing uncertainty in linguistic form. In contrast, TWD can adopt a deferred decision - making strategy for objects with insufficient information. This strategic approach enables TWD to avoid making hasty decisions that may lead to inaccurate classifications. It allows decision - makers to gather more information or conduct further analysis before reaching a conclusion, thus enhancing the reliability and accuracy of the classification results. On the other hand, normal wiggly hesitant linguistic term sets primarily concentrate on depicting uncertainty and lack an effective decision - making mechanism to deal with incomplete information, highlighting the superiority of TWD in handling complex decision - making situations.

Originating from probabilistic rough sets [3], [4], the TWD framework classifies objects into positive (POS), negative (NEG), and boundary (BND) regions to handle incomplete information. Recent extensions, such as Jia and Liu [5]'s interval-valued intuitionistic fuzzy TWD and Zhang et al. [6]'s utility-based TWD, have improved its adaptability to uncertain environments. However, these methods still face two limitations:

(1) Rigid Equivalence Relations: Existing TWD models often define equivalence classes based on crisp similarity thresholds, which fail to capture gradual transitions in complex data like air quality indices.
(2) Subjective Threshold Determination: The boundary region thresholds are typically predefined by DMs, introducing bias and reducing objectivity.

Moreover, traditional TWD frameworks neglect behavioral factors in decision-making. For example, they assume rationality in evaluating gains/losses but ignore DMs' risk preferences, such as loss aversion (e.g., preferring avoiding losses over acquiring equivalent gains). This gap is critical in real-world scenarios like air quality management, where DMs may prioritize minimizing classification errors (e.g., mislabeling unhealthy days as safe) over maximizing accuracy.

To address ambiguity in MADM, linguistic term sets (LTSs) [7] and hesitant fuzzy LTSs (HFLTSs) [8] were proposed to accommodate vague evaluations. However, HFLTSs assume equal weights for all linguistic terms, neglecting the preference differences among terms (e.g., "excellent" vs. "good"). To resolve this limitation, Pang et al. [9] introduced probabilistic linguistic term sets (PLTSs), which assign probabilities to linguistic terms to reflect DMs' hesitancy and preferences. Despite their advantages in aggregating uncertain information, PLTSs-based MADM methods [10], [11] and [12] face two critical challenges:

(1) Subjective Attribute Weighting: Existing approaches often rely on subjective weighting methods (e.g., AHP [10]) or simple normalization, ignoring data-driven objectivity. For example, Zhang et al. [10] assumed equal weights for attributes in dynamic decision-making, while Li et al. [11] used fixed weights in two-sided matching models. This subjectivity reduces the reliability of decisions, especially in complex scenarios like air quality evaluation where attribute importance varies with pollution levels.
(2) Lack of Decision-Making Mechanism: PLTSs primarily focus on information representation but lack a systematic framework for classification or decision-making. For instance, Wu et al. [12]translated qualitative assessments into interval PLTSs but did not provide a mechanism to categorize alternatives (e.g., "good" vs. "bad" air quality). This gap limits their practical application, as DMs require actionable rules rather than aggregated linguistic information.

Currently, the majority of decision-making processes rely on the presumption of complete rationality among decision-makers. Nevertheless, in real-world scenarios, subjective factors such as cognitive abilities and decision-makers' experience inevitably influence the final decision outcome. Against this backdrop, Kahneman and Tversky introduced the prospect theory in 1979, offering fresh insights into behavioral science research [13].

Wang et al. [14] proposed a TWD model based on third-generation PT, characterizing reference points through positive/negative ideal points and medians, yet overlooking the fuzziness of probabilistic linguistic information. Zhang et al. [15] integrated PT with TWD in fuzzy incomplete information systems, but relied on subjective weights and failed to address hesitation in group decision-making. Despite incorporating psychological factors, these studies have limitations in calculating conditional probabilities and ensuring objective attribute weights in probabilistic linguistic settings.

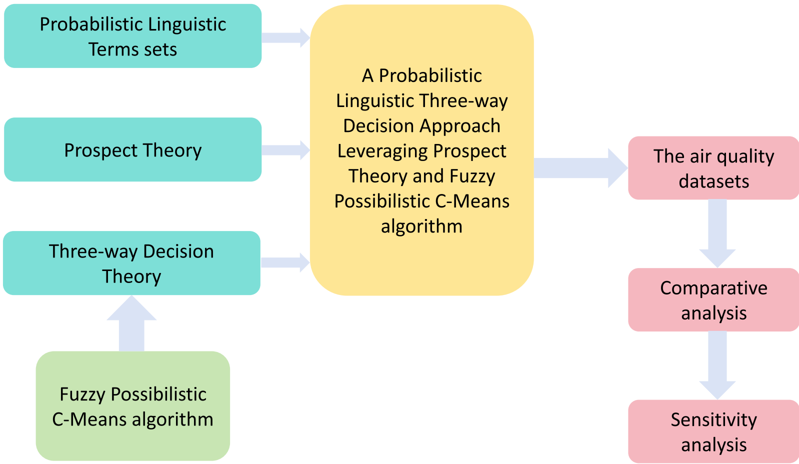

Fig. 1. The structure of this paper.

Table 1
Symbols and their corresponding explanations.

Symbols	Corresponding Explanations
$\tau+1$	Linguistic granularity in the linguistic scale
d_{is}	Distance from the data point i to the clustering center v_s
v_{sj}	Clustering center of the cluster G_s under the attribute a_j
$\mu_{G_s}(o_i)$	Membership of o_i to the G_s
$L(p)^-$	Negative ideal solution for objects
φ	Group satisfaction degree index
L_{il}^j	Happy value of the object o_j relative to o_l under the attribute a_j
G_{il}^j	Euphoria value of the object o_i relative to o_l under the attribute a_j
α,θ	Loss avoidance coefficient
λ	Risk attitude coefficient
ζ	Utility pursuit coefficient

The impetus of this study stems from the marriage of PT and PLTSs, aiming to develop a model that better recapitulates the intricate decision-making mechanisms employed by humans. Leveraging a thorough analysis of current research, this paper presents five pivotal innovations as follows.

(1) Integration of PT and TWD Models: The traditional TWD model, despite its utility in addressing uncertainty and risk-related choices, overlooks the risk preferences and decision-making procedures of individuals when confronted with potential gains and losses. Consequently, this study introduces PT into the probabilistic linguistic framework, seeking to develop a refined TWD model that aligns more closely with the genuine decision-making process of humans. By doing so, we aim to more precisely replicate the psychological behavior of decision-makers when confronted with uncertainty and risk-related choices. This integration ultimately enhances the effectiveness and reliability of decision-making outcomes.
(2) Application of the FPCM Algorithm for Processing PLTSs: In real-world scenarios, the identification of equivalence classes within multi-attribute PLTSs often poses significant challenges. To address these challenges, the present study introduces the FPCM clustering algorithm. This algorithm effectively identifies equivalence classes within PLTSs and subsequently computes conditional probabilities. This approach not only enhances classification efficiency but also bolsters the adaptability and robustness of the decision-making model.
(3) Utilizing PLTSs Evaluation Values for Attribute Weight Determination in MADM: In the context of MADM problems, the determination of attribute weights plays a pivotal role. This study departs from traditional subjective weighting methods and numerical conversions, instead directly employing the evaluation values derived from PLTSs to determine the weights of attributes. This approach takes into account the distance between an object and its NIS, alongside the overall information entropy of attributes. By doing so, it ensures the objectivity and rationality of the weight allocation, leading to more accurate and reliable decision-making outcomes.
(4) This study introduces a novel approach for constructing a group satisfaction index by integrating score functions with deviation functions. In the realm of PT, the selection of utility functions holds significant influence over decision outcomes. As an alternative to conventional utility functions, the proposed method aims to offer a more holistic representation of decision-makers' utility perception and group preferences. This advancement strives to enhance the practicality and generalizability of the decision-making model.

(5) In the realm of multi-attribute decision-making, selecting the right coefficients for the algorithm is critical. Coefficients like the loss aversion coefficient α, θ, the risk attitude coefficient λ, and the utility pursuit coefficient ζ greatly influence the decision-making outcomes. Yet, in real-world applications, these coefficient selections are often overlooked, which can introduce potential inaccuracies into the decision-making process. Consequently, this paper underscores the significance of coefficient selection and offers an in-depth analysis of how to choose these coefficients within the context of the proposed algorithm.

The structure of the following chapters in this article is outlined as follows: in Section 2, we will delve into the relevant basic theories that form the foundation of our study. This section will provide the necessary background and context for understanding the subsequent sections. Section 3 will propose the theoretical basis that underpins and supports the algorithm we have developed. This section will explain the rationale and justification for the algorithm's design. In Section 4, we will introduce the relevant theories and then apply them to an air quality dataset. The objective of this section is to verify the robustness of our algorithm and determine the optimal scale combination through a rigorous sensitivity analysis. Finally, Section 5 will serve as a conclusion, summarizing the key findings and contributions of the full text. Additionally, we have created a flow chart for this paper, depicted in Fig. 1. Next, Table 1 presents a concise compilation of the prevalent symbols utilized throughout this article.

2. Fundamental theory

In this section, we will introduce the theoretical knowledge pertaining to PLTSs, PT, and TWD involved in this article.

2.1. Probabilistic linguistic term sets

Building upon the framework of HFLTSs, Pang et al. introduced PLTSs, which integrate weights or probabilities into linguistic terms. This enhancement allows DMs to assess options with increased accuracy. A thorough review of the relevant concepts and notations associated with PLTSs can be found in reference [16]. Additionally, to enable comparisons among PLTSs, Pang et al. introduced the concepts of score functions and deviation degrees.

In the realm of PLTSs, we have a collection of linguistic terms, each assigned a probability. This collection, denoted as $L(p)$, encapsulates the essence of uncertainty in linguistic evaluations. The subscript function $\Lambda(\cdot)$ serves as a guide, mapping each linguistic term to its underlying numerical representation. The score function $\digamma(L(p))$ represents the weighted average of the linguistic terms, where the weights are given by the corresponding probabilities. This function effectively captures the 'center of gravity' of the PLTSs, providing a summary evaluation. The deviation degree $\sigma(L(p))$ quantifies the spread or dispersion of the linguistic terms within the PLTSs, giving us a sense of how diverse or concentrated the evaluations are.

To facilitate comparisons and computations, Wu et al. introduced the expected value function $\Gamma(L(p))$. This function transforms the PLTSs into a crisp, numerical value, essentially 'averaging out' the linguistic uncertainty. This allows us to leverage familiar mathematical tools and techniques for analysis, while still retaining the richness of linguistic evaluations. In a sense, it bridges the gap between qualitative linguistic assessments and quantitative numerical analysis.

Consider a set of ordered linguistic terms, S, which comprises terms ranging from s_0 to s_τ, where τ represents the total count of linguistic terms in the set. Each of these terms has a specific meaning or representation in a linguistic context, such as "very low," "low," "medium," "high," and "very high" for a rating scale.

Now, let's turn our attention to the probabilistic linguistic term set, $L(p)$. This set comprises multiple linguistic terms, each assigned a probability value. For instance, a person's assessment of a product might be represented as $L(p) = \{L_{(1)}(0.2), L_{(2)}(0.5), L_{(3)}(0.3)\}$, indicating a 20% chance of the product being "low" quality, a 50% chance of it being "medium" quality, and a 30% chance of it being "high" quality.

The subscript function, $\Lambda(\cdot)$, serves as a bridge between a linguistic term and its corresponding numerical subscript. This numerical subscript is often used to represent the term's position or level within the ordered linguistic term set S. For example, if $L_{(1)}$ stands for "low" and is the second term in S (after "very low"), then $\Lambda(L_{(1)})$ would be 1. The expected value function, $\Gamma(L(p))$, calculates the weighted average of the linguistic terms in $L(p)$ based on their respective probabilities. To do this, it multiplies each linguistic term's subscript (obtained through the subscript function) by its probability and then sums these values. This sum is then divided by the total probability of all terms in $L(p)$ to obtain the expected value. In essence, it provides a numerical representation of the overall linguistic assessment, considering the probabilities assigned to each possible term.

2.2. Three-way decision

Yao [3] introduced a novel TWD model, where the loss function serves as a metric for risk assessment. Subsequently, Zhang et al. [17] built upon this foundation by integrating utility theory to enhance classification accuracy. Under the umbrella of decision-theoretic rough sets theory [4], the condition of an object, denoted as o, is categorized as either "good" (χ) or "bad" ($\neg\chi$). Correspondingly, three decision actions are employed: accept (($\flat_P$), defer (($\flat_B$), and reject (($\flat_N$). These actions directly correspond to the positive region (POS), boundary region (BND), and negative region (NEG) of the rough set, respectively.

When an object assumes the "good" state χ, executing distinct decision actions ($\flat_P$, $\flat_B$, and $\flat_N$) generates corresponding utility functions ($\flat_P$, $\flat_{BP}$, and $\flat_{NP}$). Analogously, in the "bad" state $\neg\chi$, similar utility functions (μ_{PN}, μ_{BN}, and μ_{NN}) emerge. Furthermore, the conditional probabilities of an object o belonging to either state χ and $\neg\chi$ are designated as $Pr(\chi|[o])$ and $Pr(\neg\chi|[o])$, respectively, with their combined sum totaling 1. Drawing upon the aforementioned definitions, the expected utility $\Psi(\flat_\star|[o])$ ($\star = P, B, N$) of an object o under the three decision actions can be calculated using the formula:

$$\Psi(\flat_\star|[o]) = \mu_{\star P} Pr(\chi|[o]) + \mu_{\star N} Pr(\neg\chi|[o]). \tag{1}$$

This formula serves to quantify the anticipated utility that object o aims to attain across the range of decision actions.

2.3. Prospect theory

In the realm of risk decision-making, individuals tend to exhibit limited rationality, deviating from the ideal of complete rationality. Consequently, the decision-making process does not always align with the principles of expected utility theory. Kahneman and his colleagues introduced the prospect theory in 1979, emphasizing that in uncertain decision-making scenarios, the outcomes are shaped by the preferences and biases of the decision-makers. This implies that different individuals may arrive at varied decisions for the same problem. The prospect value function, defined as $V = \sum_{t=1}^{t} \omega(p_t)v(x_t)$, incorporates a decision weight function $\omega(p_t) = \frac{p_t^{\xi}}{\left(p_t^{\xi}+(1-p_t)^{\xi}\right)^{1/\xi}}$ and a value function $v(x_t)$ with the following form:

$$v(x_t) = \begin{cases} x^{\theta},\ x > 0, \\ -\lambda(-x)^{\alpha},\ x < 0. \end{cases} \tag{2}$$

Here, ξ, ∂, σ represent risk attitude coefficients, while λ signifies the loss avoidance factor. The prospect theory provides a framework to understand how individuals evaluate risks and make decisions in uncertain environments.

3. Theory of the probabilistic linguistics TWD model based on PT

This section will elaborate on the theoretical framework on which the construction of this algorithm is relied upon.

3.1. Enhance approach for deriving the PLTSs matrix

In the context of multi-expert and multi-attribute decision-making scenarios, it is crucial to acknowledge that diverse DMs often provide varying evaluation information due to their unique professional backgrounds, accumulated experience, and positional perspectives. Here is a concise overview of the pertinent elements:

(1) Set $O = \{o_i | i = 1, 2, \dots, n, i \in N\} (n \geq 2)$ represents a non-empty finite set of objects containing n objects, which is called the object set.

(2) $A = \{a_j | j = 1, 2, \dots, m, j \in M\} (m \geq 2)$ designates a non-empty finite set of m attributes, known as the attribute set.

(3) $W = (\omega_1, \omega_2, \dots, \omega_m)^T$ represents the weight vector of attributes, where ω_j represents the importance of the j^{th} attributes, satisfy $\omega_j \geq 0$ and $\sum_{j=1}^{m} \omega_j^2 = 1$.

(4) $H = \{h_t | t = 1, 2, \dots, g, t \in G\} (g \geq 1)$ denotes a non-empty finite set of g DMs, referred to as the DM set. In this scenario, each DM h_t may provide different evaluations for the objects o_i based on the attributes a_j, reflecting their individual perspectives and expertise.

To elaborate on the provided definition, let us consider a scenario where a group of DMs is involved in a collective assessment process. This group, denoted as H, comprises g individual DMs, each associated with a weight that reflects their importance or influence in the process. The collection of these weights is represented by the vector $\left(\epsilon^{(1)}, \epsilon^{(2)}, \dots, \epsilon^{(g)}\right)'$, where the sum of all weights is equal to 1. This ensures that the weights are properly normalized and can be used as relative measures of importance.

Within this group, a subset of g' DMs contributes LTSs to the assessment. Each LTSs, denoted as $S^{(t)}$ for the t-th DM, comprises a collection of linguistic terms $s_\gamma^{(t)}$ that range from $\gamma = 0$ to $\gamma = \tau$. These linguistic terms represent qualitative evaluations or assessments made by the individual DMs. The remaining $g - g'$ DMs do not provide any assessments and are therefore not considered in the collective evaluation.

To combine the assessments of the participating DMs, a probabilistic linguistic term set is constructed. This probabilistic linguistic term set, denoted as $L(p)$, is a collection of linguistic terms $L^{(k)}$ that are weighted by their probabilities $p^{(k)}$. The probability $p^{(k)}$ of a linguistic term $L^{(k)}$ is calculated by summing the weighted frequencies of that term across all contributing DMs. Specifically, the frequency $\aleph\left(L^{(k)}\right)^{(t)}$ of $L^{(k)}$ for the t-th DM is multiplied by the corresponding weight $\epsilon^{(t)}$ and then summed over all contributing DMs ($t = 1, 2, \dots, g'$).

If the weights of the DMs are equal or unspecified, a common practice is to assume that each DM has an equal weight of $1/g$. This ensures that all DMs contribute equally to the collective assessment, regardless of their individual expertise or influence. In summary, the provided definition describes a method for aggregating linguistic assessments from a group of decision makers into a probabilistic linguistic term set that represents the collective evaluation of the group. The weighting scheme allows for considering the relative importance of different DMs, while the linguistic terms provide a qualitative representation of the assessments.

When synthesizing the evaluation data from each Decision Maker (DM) based on the probability of a linguistic term, the ultimate probabilistic linguistic decision matrix, denoted as $\overline{\mathfrak{R}} = |\overline{L_{ij}}(p)|_{n\times m}$, is structured as follows:

$$\overline{\Re}=\begin{bmatrix}\overline{L_{11}}(p) & \overline{L_{12}}(p) & \cdots & \overline{L_{1m}}(p)\\ \overline{L_{21}}(p) & \overline{L_{22}}(p) & \cdots & \overline{L_{2m}}(p)\\ \vdots & \vdots & \ddots & \vdots\\ \overline{L_{n1}}(p) & \overline{L_{n2}}(p) & \cdots & \overline{L_{nm}}(p)\end{bmatrix}.$$

Each Probabilistic Linguistic Term Set within $\overline{\Re}$ undergoes normalization according to the formula (6). Specifically, for the ith object under the jth attribute:

$$L_{ij}^{(k)}\left(p_{ij}^{(k)}\right)=\begin{cases}\overline{L_{ij}^{(k)}}\left(p_{ij}^{(k)}\right), & a_j \text{ is a benefit attribute,}\\ \operatorname{neg}\left(\overline{L_{ij}^{(k)}}\left(p_{ij}^{(k)}\right)\right), & a_j \text{ is a cost attribute.}\end{cases} \tag{3}$$

Moreover, the probabilistic linguistic decision matrix $\overline{\Re}=|\overline{L_{ij}}(p)|_{n\times m}$ is then normalized to $\Re=|L_{ij}(p)|_{n\times m}$, where $L_{ij}(p)=\left\{L_{ij}^{(k)}(p_{ij}^{(k)})|k=1,2,\ldots,\#L_{ij}(p),p_{ij}^{(k)}>0,\sum_{k=1}^{\#L_{ij}(p)}p_{ij}^{(k)}=1\right\}$.

3.2. Optimal attribute weight estimation

A probabilistic linguistic decision matrix, denoted by $\Re=|L_{ij}(p)|_{n\times m}$, essentially encapsulates the linguistic evaluations and their corresponding probabilities for various alternatives across multiple criteria. Each cell $L_{ij}(p)$ in this matrix represents a set of linguistic terms, such as "very good," "good," "average," and so on, along with their associated probabilities. These probabilities reflect the degree of certainty or uncertainty associated with each linguistic evaluation.

Now, let's turn our attention to the concept of the negative ideal solution (NIS). The NIS represents a theoretical benchmark that is constructed based on the worst-case linguistic evaluations across all criteria. Mathematically, this is denoted as $L(p)^-=(L_1(p)^-,L_2(p)^-,\ldots,L_m(p)^-)$. To determine the NIS, we examine each criterion j independently. For each criterion, we find the linguistic term that has the lowest product of its probability and numerical representation (as determined by the subscript function $\Lambda(\cdot)$). This lowest-valued linguistic term, along with its associated probability, forms the component $L(p)_j^-$ of the NIS for that particular criterion. By doing this for all criteria, we arrive at the complete NIS, which represents the least desirable outcome in terms of linguistic evaluations and their probabilities.

Acknowledging that DMs tend to be more sensitive to losses, the approach in this article defines the weighted deviation as the difference between an object and its NIS. The total weighted deviation of all objects from the NIS across all attributes is expressed as:

$$D(\omega)=\sum_{j=1}^{m}\sum_{i=1}^{n}\omega_j d\left(L_{ij}(p),L_j(p)^-\right). \tag{4}$$

Here, ω_j represents the weight associated with the j-th attribute, reflecting its relative importance in the decision-making process. The function $d\left(L_{ij}(p),L_j(p)^-\right)$ measures the deviation between the linguistic term $L_{ij}(p)$ for object i and attribute j and the corresponding linguistic term in the NIS, $L_j(p)^-$. This deviation metric captures how far an object's performance is from the "worst" possible performance defined by the NIS. By summing these weighted deviations across all objects and attributes, we obtain $D(\omega)$ which represents the overall deviation of the considered objects from the NIS.

The Euclidean distance between two probabilistic linguistic term sets, denoted as $L_1(p)$ and $L_2(p)$, serves as a quantitative measure of the dissimilarity between them. Both sets are composed of linguistic terms associated with respective probabilities. Specifically, $L_1(p)$ comprises $\#L(p)_1$ linguistic terms $L_1^{(k)}$ with probabilities $p_1^{(k)}$ for each k from 1 to $\#L(p)_1$, while $L_2(p)$ has a similar structure with $\#L(p)_2$ terms $L_2^{(k)}$ and probabilities $p_2^{(k)}$ for each k from 1 to $\#L(p)_2$. It is important to note that in this context, $\#L(p)_1$ is equal to $\#L(p)_2$, ensuring a one-to-one comparison between the two sets.

The Euclidean distance between $L_1(p)$ and $L_2(p)$ is calculated by taking the square root of the sum of squared differences between the weighted linguistic terms. Here, the weight of each linguistic term is determined by multiplying its associated probability with a function Λ, which typically represents a mapping from linguistic terms to numerical values. The sum is then divided by the number of terms in $L_1(p)$ (or equivalently, $L_2(p)$ since they have the same number of terms) to normalize the distance measure. This normalized Euclidean distance provides a quantitative assessment of how dissimilar the two probabilistic linguistic term sets are, with a lower value indicating a higher degree of similarity.

The probabilistic linguistic entropy $E(L(p))$ serves as a measure of the uncertainty associated with a given probabilistic linguistic term set $L(p)$. Consider a linguistic term set S that comprises a range of linguistic terms s_γ from $\gamma=0$ to $\gamma=\tau$. The PLTS $L(p)$ represents these linguistic terms in a probabilistic manner, where each linguistic term $p^{(k)}$ is associated with a probability distribution. The number of distinct probabilistic linguistic terms in $L(p)$ is denoted by $\#L(p)$ and equals K.

The computation of $E(L(p))$ hinges on two fundamental concepts: the hesitant fuzzy linguistic entropy, which gauges the extent of fuzziness or uncertainty inherent in the linguistic terms, and the probabilistic linguistic equivalence transformation function ϱ. This transformation function allows for a mapping between linguistic terms and their probabilistic representations, taking into account the inherent uncertainty in linguistic expressions.

By combining these two concepts, the probabilistic linguistic entropy $E(L(p))$ provides a quantitative assessment of the overall uncertainty within the PLTSs. A higher value of $E(L(p))$ indicates a greater degree of uncertainty, while a lower value suggests a more definitive or precise representation of the linguistic terms. This measure is particularly useful in scenarios where linguistic data is subject to vagueness, ambiguity, or incompleteness, enabling analysts to make informed decisions based on a quantitative understanding of the underlying uncertainty.

$$E(L(p)) = \frac{1}{K(\sqrt{2}+1)}\left(\sin\frac{\pi p^{(k)}}{2} + \cos\frac{\pi p^{(k)}}{2} + \sin A + \cos\left(\frac{\pi}{2} - A - 1\right)\right), \tag{5}$$

where $A = \frac{\pi(\varrho(L^{(k)})+\varrho(L^{(K-k+1)}))}{2}$, and Probabilistic language equivalent transformation function ϱ is defined as:

$$\varrho(L(p)) = \begin{cases} \frac{\Lambda(L^{(k)})}{\tau}p^{(k)}, & \text{if } L^{(k)} \in L(p), \\ 0, & \text{otherwise}, \end{cases}$$

and $\varrho : [0,\tau] \to [0,1]$.

Based on the definition of weighted deviation, an elevated deviation value between an object and its NIS indicates a more favorable performance of that object. Consequently, during the evaluation process, it is crucial to incorporate the metric of information entropy, which provides a quantitative measure of the uncertainty associated with information. Furthermore, we can calculate the total information entropy for attribute a_j. Notably, a lower value of evaluation information entropy for a specific attribute suggests that the attribute is information-rich, thereby playing a significant role among the entire set of attributes. Given this insight, assigning higher weight values to such attributes is crucial to ensure the accuracy and fairness of the evaluation results. By doing so, we can ensure that the attributes that contribute the most information are given due consideration in the evaluation process.

When the total weighted deviation ($D(\omega)$) and the attribute's total information entropy (a_j) reach their respective extrema, optimal attribute weights are attained. A multiobjective programming model is formulated to maximize ($D(\omega)$) and minimize the entropy ($\sum_{i=1}^{n} E(L_{ij}(p))$) (equivalently, maximize ($\sum_{i=1}^{n}(1 - E(L_{ij}(p)))$)) to determine attribute weights. Constraints ensure the squared sum of weights equals 1, and all weights are non-negative. The model is expressed as:

$$\begin{cases} \max & \sum_{j=1}^{m}\sum_{i=1}^{n}\omega_j d(L_{ij}(p), L_j(p)^-), \\ \max & \sum_{j=1}^{m}\omega_j\left[\sum_{i=1}^{n}(1 - E(L_{ij}(p)))\right], \\ \text{s.t.} & \sum_{j=1}^{m}\omega_j^2 = 1, \\ & \omega_j \ge 0, \forall j = 1,2,\dots,m. \end{cases} \tag{6}$$

In the following, formula (6) is converted into the following form:

$$\begin{cases} \max \sum_{j=1}^{m}\omega_j\left(\sum_{i=1}^{n} d\left(L_{ij}(p), L_j(p)^-\right) + \sum_{i=1}^{n}\left(1 - E\left(L_{ij}(p)\right)\right)\right) \\ \text{s.t.} \sum_{j=1}^{m}\left(\omega_j\right)^2 = 1 \ \omega_j \ge 0, \quad j = 1,2,\dots,m. \end{cases} \tag{7}$$

Using the Lagrange function and taking into account the normalization constraint imposed on the attribute weights, the expression for calculating the attribute weight ω_j is derived as follows in accordance with Formula (11):

$$\omega_j = \frac{\sum_{i=1}^{n}\left[d(L_{ij}(p), L_j(p)^-) + 1 - E(L_{ij}(p))\right]}{\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\left[d(L_{ij}(p), L_j(p)^-) + 1 - E(L_{ij}(p))\right]\right)}. \tag{8}$$

3.3. Innovative index for assessing group satisfaction

This paper describes a novel method for assessing the level of satisfaction with PLTSs, focusing on the concept of scoring functions and bias degrees.

Definition 1. In order to compute the group satisfaction degree index, both the score function $\digamma(L(p))$ and the average deviation degree $\overline{\alpha}(L(p))$ are simultaneously taken into account, thereby effectively utilizing DM information. The score function $\digamma(L(p))$ effectively captures the overall characteristics of $L(p)$ within the PLTSs.

Let $\mathcal{L} = \{s,\tau\}$ be a linguistic term set, and let $L(p) = \{L^{(k)}(p^{(k)})|k = 1,2,\dots,\#L(p)\}$ be a probabilistic linguistic term set. Then the following expression:

$$\varphi(L(p)) = \frac{\Lambda(\digamma(L(p)))}{\tau + \overline{\sigma}(L(p))} = \frac{\Lambda(\digamma(L(p)))}{\tau + \frac{1}{|L(p)|}\sum_{k=1}^{|L(p)|} p^{(k)}\left|\Lambda(L^{(k)}(p^{(k)})) - \Lambda(\digamma(L(p)))\right|}, \tag{9}$$

is referred to as the group satisfaction degree index in PLTSs. Here, $\Lambda(\cdot)$ is the subscript function that maps linguistic terms to numerical values, $\digamma(L(p))$ is the score function of the PLTSs calculated according to Equation (10), and $\overline{\sigma}(L(p))$ is the average deviation function of the PLTSs, which quantifies the degree of divergence or dispersion among the decision groups represented by the PLTSs.

$$\digamma(L(p)) = s_{\overline{\alpha}}, \quad \overline{\alpha} = \sum_{k=1}^{\#L(p)}\Lambda(L^{(k)})p^{(k)} \Big/ \sum_{k=1}^{\#L(p)} p^{(k)},$$

$$\sigma(L(p)) = \left(\sum_{k=1}^{\#L(p)} \left(p^{(k)} \left(\Lambda(L^{(k)}) - \overline{\alpha} \right) \right)^2 \right)^{1/2} \Big/ \sum_{k=1}^{\#L(p)} p^{(k)}. \tag{10}$$

$$\Gamma(L(p)) = \sum_{k=1}^{\#L(p)} \left(\frac{\Lambda(L^{(k)}) p^{(k)}}{\tau} \right) \Big/ \sum_{k=1}^{\#L(p)} p^{(k)}. \tag{11}$$

To compute the group satisfaction degree index, both the score function $F(L(p))$ and the average deviation degree $\overline{\sigma}(L(p))$ are considered simultaneously, thereby effectively utilizing decision-maker (DM) information. The score function $F(L(p))$ effectively captures the overall characteristics of $L(p)$ within the PLTSs. The higher the value of $F(L(p))$ is, the greater the satisfaction level is likely to be. Concurrently, the average deviation degree $\overline{\alpha}(L(p))$ reflects the extent of disagreement among decision groups. A lower value of $\overline{\alpha}(L(p))$ indicates a higher degree of consensus among groups, which subsequently translates into a higher satisfaction degree.

3.4. Conditional probability estimation via the FPCM

Within the domain of TWD, the precise computation of conditional probability occupies a central role in developing scientifically rigorous models. Prior studies have witnessed scholars defining conditional probability primarily through the examination of equivalence or dominance relations among objects. Nevertheless, the complexity of datasets often renders rigid categorization of objects into clusters inflexible and prone to inaccuracies. To overcome this limitation, the advent of the innovative fuzzy clustering algorithm, FPCM, has been significant. This algorithm seamlessly integrates fuzzy theory, encompassing both the distance between data points and cluster centers and uncertainty factors. By optimizing the algorithm to minimize both distance and uncertainty, it accurately quantifies the membership degree of each data point to various clusters, assigning precise weights to each object and cluster. In this section, we intend to utilize this groundbreaking clustering algorithm to automatically categorize objects in a probabilistic language setting, thereby facilitating the accurate calculation of conditional probability. This, ultimately, provides robust foundational support for the construction of our models.

Firstly, we shall present the FPCM algorithm, which is outlined as follows.

Definition 2. Let $\{Z_1, Z_2, Z_3, \ldots, Z_n\}$ be a set of n data points represented by p-dimensional feature vectors $z_j = [z_{1j},\ z_{2j},\ \ldots,\ z_{pj}]^{\mathrm{T}} \in \mathcal{R}^p$. The $p \times n$ data matrix Z has the cluster matrix $A = \left[a_1\ ,\ a_2\ , \ldots,\ a_c\right]$ $(1 < c < \mathbf{n})$ and the membership matrix $U \equiv \lceil \mu_{ij} \rfloor_{cxn}$ where μ_{ij} is the membership value of z_j belonging to a_i. Let $S = [s_{il}]_{cxc}$ represent the weighting matrix, with a_{il} being the weighting value between a_i and a_l. The fuzzy exponent f is greater than 1.

The proposed objective function is:

$$J^{f}_{FPCM-S}(U,T,\Sigma,Z) = \sum_{i=1}^{c} \sum_{j=1}^{n} (\mu_{ij}^{f} + t_{ij}^{\delta}) \left[(\underline{Z_j} - \underline{a_i})' (\underline{Z_j} - \underline{a_i}) \right] - \frac{1}{c(c-1)} \sum_{i=1}^{c} \sum_{l=1}^{c} s_{il} (\underline{a_i} - \underline{a}l)' (\underline{ai} - \underline{a_l}). \tag{12}$$

Constraints:

(1) Membership: $\sum_{i=1}^{r} \mu_{ij} = 1\ ,\ ^{\vee} j = 1, 2, \ldots, n.$
(2) Typicality: $\sum_{j=1}^{n} t_{ij} = 1\ ,\ \forall i = 1, 2, \ldots, c.$
(3) Attribute weight: $\sum_{j=1}^{m} \omega_j^2 = 1\ ,\ \omega_j \geq 0, \forall j = 1, 2, \cdots, m.$

Where S_{ij} is defined as:

$$S_{il} = \frac{\left[\left| \bar{z} - a_i \right|^2 + \left| \bar{z} - a_l \right|^2 \right] - \min_{1 \leq r,s \leq c} \left[\left| \bar{z} - a_r \right|^2 + \left| \bar{z} - a_s \right|^2 \right]}{\max_{1 \leq r,s \leq c} \left[\left| \bar{z} - a_r \right|^2 + \left| \bar{z} - a_s \right|^2 \right] - \min_{1 \leq r,s \leq c} \left[\left| \bar{z} - a_r \right|^2 + \left| \bar{z} - a_s \right|^2 \right]}. \tag{13}$$

And defined $\hat{\mu}_{il}$ as follows:

$$\hat{\mu}_{il} = \beta \cdot \frac{\min \left\{ \left| \bar{Z} - a_i \right|^2, \left| \bar{Z} - a_l \right|^2 \right\}}{\max \left\{ \left| \bar{Z} - a_i \right|^2, \left| \bar{Z} - a_l \right|^2 \right\}} \quad \text{with } 0 \leq \beta \leq 1. \tag{14}$$

In the context of the dataset detailed in this paper, the implementation of the FPCM algorithm involves establishing the original matrix as Z, with the elements defined as $z_{ij} = \sum_{k=1}^{\#L(p)} \left(\frac{\Lambda\left(L_{ij}^{(k)}\right) p_{ij}^{(k)}}{\tau} \right) / \sum_{k=1}^{\#L(p)} p_{ij}^{(k)}$. In terms of commonality, we have established the fuzzy factor m as equal to 2.

We allocate each object to a particular cluster by determining the highest membership degree for each object, as described by the following equation:

$$\mu_{G_s}(o_i) = \max_{1 \leq s \leq C} \mu_{is}, \quad i = 1, 2, \ldots, n. \tag{15}$$

In this equation, G_s represents the cluster to which the object o_i is assigned, and $\mu_{G_s}(o_i)$ indicates the membership degree of the object o_i belonging to the cluster G_s. Consequently, following the aforementioned process, the n objects are categorized into C distinct clusters, namely, $G_1, G_2, \ldots, G_C$. These C clusters can be interpreted as C equivalence classes, denoted as $[o]_1, [o]_2, \ldots, [o]_C$. Thus, the task of identifying equivalence classes essentially consists of finding the object with the maximum membership degree in each class.

The FPCM algorithm is essential for subsequent research and plays a crucial role in processing multi-attribute PLTSs data. Its significance is particularly pronounced in the upcoming air quality assessment experiments. Air quality data from Chinese cities is highly complex and diverse, yet the FPCM algorithm can effectively analyze and mine this data. By calculating the distance between data points and cluster centers and taking uncertainty factors into account, the FPCM algorithm clusters air quality data from cities like Beijing, Shanghai, and Guangzhou. This process groups data points with similar air quality characteristics into the same equivalence class. The precise division of these classes provides a robust foundation for subsequent conditional probability calculations. Conditional probability is pivotal in constructing scientific decision-making models, as it enables decision-makers to clearly understand the likelihood of various decisions under different air quality conditions. This understanding is instrumental in developing reasonable air quality management strategies. Only with accurate conditional probabilities can we construct efficient and reliable decision-making models for the scientific assessment and effective management of air quality.

After obtaining the set of equivalence classes $\Theta = \{[o]_1, [o]_2, \ldots, [o]_C\}$ and given that the "good state"; χ is known, we can extend Zadeh's classical conditional probability formula to compute the conditional probability for the equivalence class $[o]_s$, which is formulated as follows:

$$Pr(\chi|[o]_s) = \frac{\sum_{o_i \in \chi \cap [o]_s} \mu_{G_s}(o_i)}{\sum_{o_i \in [o]_s} \mu_{G_s}(o_i)}, \text{where } i = 1, 2, \ldots, n, \text{and } s = 1, 2, \ldots, C. \tag{16}$$

In this context, $\mu_{G_s}(o_i)$ denotes the degree of membership of o_i pertaining to the class $[o]_s$.

3.5. Integration of TWD in PT analysis

In this section, I will detail the combination of PT and TWD in this paper.

The DM will carefully evaluate the utility from the selected object and juxtaposition it with the utility generated by the other objects. If the evaluation reveals that the selected object yields superior outcomes, the DM will be pleased. Conversely, in the event of inferior results, the DM will express regret. In this article, the group satisfaction index degree $\varphi(L(p))$ defined in Section 3.3 serves as a surrogate for the utility function u associated with the object. Consequently, pertaining to the attribute a_j, the happy value of the object o_i relative to another object o_l is formulated as follows:

$$L_{il}^{j} = \begin{cases} (-\lambda)(\varphi(L_{ij}(p)) - \varphi(L_{lj}(p)))^{\alpha}, \\ \text{if } \varphi(L_{ij}(p)) < \varphi(L_{lj}(p)), \\ 0, \text{if } \varphi(L_{ij}(p)) \geq \varphi(L_{lj}(p)). \end{cases} \tag{17}$$

This formulation ensures a rigorous and objective comparison of the satisfaction levels associated with different objects, thereby facilitating informed decision-making.

The euphoria value of the object o_i relative to o_l is denoted as follows:

$$G_{il}^{j} = \begin{cases} 0, & \text{if} \varphi(L_{ij}(p)) < \varphi(L_{lj}(p)), \\ (\varphi(L_{ij}(p)) - \varphi(L_{lj}(p)))^{\theta}, & \text{if} \varphi(L_{ij}(p)) \geq \varphi(L_{lj}(p)). \end{cases} \tag{18}$$

In the process of decision-making, disparities in physical dimensions can exert diverse impacts on the ultimate outcome. To uphold the uniformity and precision of our decisions, it is imperative to standardize the pertinent matrix data. Therefore, we have adopted a normalization method. Specifically, for the happiness matrix $L^j = (L_{il}^j)_{n\times n}$ and excitement matrix $G^j = (G_{il}^j)_{n\times n}$, the corresponding normalized matrices are derived as $\overline{L^j} = (\overline{L_{il}^j})_{n\times n}$ and $\overline{G^j} = (\overline{G_{il}^j})_{n\times n}$. The normalization process adheres to the following mathematical formula:

$$\begin{aligned} &\overline{L^j} = L_{il}^j / LG^{\max}, \\ &\overline{G^j} = G_{il}^j / LG^{\max}. \end{aligned} \tag{19}$$

Here, $LG^{\max}$ denotes the maximum value among all happiness and excitement indices.

Integrating all pertinent attributes, the Eq. (19) presented hereinafter is employed to compute the aggregate happy value $L(o_i)$ and the euphoria value $G(o_i)$ of the object o_i in comparison with other objects:

$$\begin{aligned} &L(o_i) = \sum_{l=1}^{n} \sum_{j=1}^{m} \omega_j L_{il}^j, \quad i = 1, 2, \ldots, n, \\ &G(o_i) = \sum_{l=1}^{n} \sum_{j=1}^{m} \omega_j G_{il}^j, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{20}$$

The formula for calculating the perception value of object utility is as follows:

$$F(o_i) = L(o_i) + G(o_i), \quad i = 1, 2, \ldots, n. \tag{21}$$

The ranking of objects will be established based on the descending order of the $F(o_i)$ value. Additionally, to reduce the influence of dimensions on subsequent outcomes, we normalize the utility perception value $F(o_i)$ to $\overline{F}(o_i)$ using the formula provided below:

$$\overline{F}(o_i) = \frac{F(o_i) - \min\left\{F(o_i)\right\}}{\max\left\{F(o_i)\right\} - \min\left\{F(o_i)\right\}}, \quad i = 1, 2, \ldots, n. \tag{22}$$

Upon acquiring the equivalence class C in Section III-D, the average utility perception value $\mathcal{F}([o]_s)$ for each equivalence class $[o]_s$ can be derived as follows:

$$\mathcal{F}([o]_s) = \frac{\sum_{o_i \in [o]_s} \overline{F}(o_i)}{|o_i|}, \quad i = 1, 2, \ldots, n, \quad s = 1, 2, \ldots, C. \tag{23}$$

Drawing inspiration from the utility-based TWD model and incorporating the concept of utility perception value, we formulate relative utility functions that are grounded in the normalized utility perception value $\mathcal{F}([o]_s)$ pertaining to each equivalence class $[o]_s$.

A utility pursuit coefficient, denoted as ζ and ranging from 0 to 1, is introduced to quantitatively assess the relative utility functions associated with the action $\flat_B$ under various states. This coefficient serves as a metric to explain the degree of preference exhibited by decision-makers (DMs) towards postponing decision-making. Furthermore, in the context where an object belongs to the set $\chi(P)$, the utility function corresponding to $\flat_N$ is employed as the benchmark. Relative to this benchmark, the relative utility functions of $\flat_P$, $\flat_B$, and $\flat_N$ with respect to $\flat_N$ are computed as follows:

$$\begin{aligned}
&\widehat{\mu_{PP}} = \mathcal{F}([o]_s) - \min\{\overline{F}(o_i)\},\\
&\widehat{\mu_{BP}} = \zeta \cdot (\mathcal{F}([o]_s) - \min\{\overline{F}(o_i)\}),\\
&\widehat{\mu_{NP}} = 0.
\end{aligned} \tag{24}$$

In the alternative scenario where an object does not belong to $\neg\chi(N)$, the utility function associated with $\flat_P$ serves as the benchmark. Relative to this benchmark, the relative utility functions of $\flat_P$, $\flat_B$, and $\flat_N$ with respect to $\flat_P$ are defined as:

$$\begin{aligned}
&\widehat{\mu_{PN}} = 0,\\
&\widehat{\mu_{BN}} = \mathcal{F}([o]_s) - \min\{\overline{F}(o_i)\},\\
&\widehat{\mu_{NN}} = \zeta \cdot (\mathcal{F}([o]_s) - \min\{\overline{F}(o_i)\}).
\end{aligned} \tag{25}$$

It is noteworthy that, as detailed in Section II-B, the following inequality relations persist:

$$0 = \widehat{\mu_{NP}} \le \widehat{\mu_{BP}} \le \widehat{\mu_{PP}} \quad \text{and} \quad 0 = \widehat{\mu_{PN}} \le \widehat{\mu_{BN}} \le \widehat{\mu_{NN}}.$$

Based on our prior deliberations, leveraging the conditional probability and the relative utility functions detailed in Sections III-D and III-E, we can calculate the anticipated utility values, denoted as $\Psi(\flat_\star|[o]_s)$ ($\star = P, B, N$), for any given equivalence class $[o]_s$ of an object, with respect to three distinct actions. These calculations are performed using the following formulas:

$$\Psi(\flat_P|[o]_s) = \widehat{\mu_{PP}} \cdot Pr(\chi|[o]_s), \tag{26}$$

$$\Psi(\flat_B|[o]_s) = \widehat{\mu_{BP}} \cdot Pr(\chi|[o]_s) + \widehat{\mu_{BN}} \cdot Pr(\neg\chi|[o]_s), \tag{27}$$

$$\Psi(\flat_N|[o]_s) = \widehat{\mu_{NN}} \cdot Pr(\neg\chi|[o]_s). \tag{28}$$

Based on the Bayesian method, the behavior with the highest expected utility value should be selected. Therefore, for each equivalence class, we can derive the following three decision rules:

(P') If $\Psi(\flat_P|[o]_s) \ge \Psi(\flat_B|[o]_s)$ and $\Psi(\flat_P|[o]_s) \ge \Psi(\flat_N|[o]_s)$,

it shall be determined that $[o]_s \in POS(\chi)$

(B') If $\Psi(\flat_B|[o]_s) \ge \Psi(\flat_P|[o]_s)$ and $\Psi(\flat_B|[o]_s) \ge \Psi(\flat_N|[o]_s)$,

it shall be determined that $[o]_s \in BND(\chi)$

(N') If $\Psi(\flat_N|[o]_s) \ge \Psi(\flat_P|[o]_s)$ and $\Psi(\flat_N|[o]_s) \ge \Psi(\flat_B|[o]_s)$,

it shall be determined that $[o]_s \in NEG(\chi)$.

The decision rules (P1') – (N1') can be equivalently expressed as follows:

(P1') If $Pr(\chi|[o]_s) \geq \widehat{\alpha}_s$ and $Pr(\chi|[o]_s) \geq \widehat{\eta}_s$,

then classify $[o]_s$ as belonging to the positive set $POS(\chi)$

(B1') If $Pr(\chi|[o]_s) \leq \widehat{\alpha}_s$ and $Pr(\chi|[o]_s) \geq \widehat{\beta}_s$,

then classify $[o]_s$ as belonging to the boundary set $BND(\chi)$

(N1') If $Pr(\chi|[o]_s) \leq \widehat{\eta}_s$ and $Pr(\chi|[o]_s) \leq \widehat{\beta}_s$,

then classify $[o]_s$ as belonging to the negative set $NEG(\chi)$

where the thresholds $\widehat{\alpha}_s$, $\widehat{\beta}_s$, and $\widehat{\eta}_s$ are computed using the following formulas:

$$\begin{aligned}
\widehat{\alpha}_s &= \frac{\widehat{\mu_{BN}}^s}{\widehat{\mu_{BN}}^s + (\widehat{\mu_{PP}}^s - \widehat{\mu_{BP}}^s)} = \frac{\zeta \cdot \widehat{\mu_{NN}}^s}{\zeta \cdot \widehat{\mu_{NN}}^s + (1-\zeta) \cdot \widehat{\mu_{PP}}^s}, \\
\widehat{\beta}_s &= \frac{\widehat{\mu_{NN}}^s - \widehat{\mu_{BN}}^s}{(\widehat{\mu_{NN}}^s - \widehat{\mu_{BN}}^s) + \widehat{\mu_{BP}}^s} = \frac{(1-\zeta) \cdot \widehat{\mu_{NN}}^s}{(1-\zeta) \cdot \widehat{\mu_{NN}}^s + \zeta \cdot \widehat{\mu_{PP}}^s}, \\
\widehat{\eta}_s &= \frac{\widehat{\mu_{NN}}^s}{\widehat{\mu_{NN}}^s + \widehat{\mu_{PP}}^s}.
\end{aligned} \tag{29}$$

Proposition 1. *Within the context of TWD, decision outcomes are produced exclusively when $\zeta > 0.5$.*

Proof. Evidence supporting this assertion is derived from the decision rule (P1') - (N1'), which stipulates that, in the presence of a boundary domain, the TWD necessitates $\hat{\alpha}_s > \hat{\beta}_s$. Consequently, based on Eq. (29), it is established that $\widehat{\mu_{PP}}^s > 0$, $\widehat{\mu_{NN}}^s > 0$, and $\zeta \in [0,1]$. Subsequent derivation proceeds as follows:

$\widehat{\alpha}_s > \widehat{\beta}_s \Leftrightarrow \frac{\zeta \widehat{\mu_{NN}}^s}{\zeta \widehat{\mu_{NN}}^s + (1-\zeta)\widehat{\mu_{PP}}^s} - \frac{(1-\zeta)\widehat{\mu_{NN}}^s}{(1-\zeta)\widehat{\mu_{NN}}^s + \zeta \widehat{\mu_{PP}}^s} > 0 \Leftrightarrow 2\zeta - 1 > 0 \Leftrightarrow \zeta > 0.5.$

We can deduce that $\widehat{\beta}_s < \widehat{\eta}_s < \widehat{\alpha}_s$ holds true. Consequently, the decision rules (P1') – (N1') can be streamlined into (P2') – (N2') as detailed below:

(P2') If the probability of χ given $[o]_s$ is at least $\widehat{\alpha}_s$,

then classify $[o]_s$ as belonging to the positive set of χ.

(B2') If the probability of χ given $[o]_s$ falls between $\widehat{\beta}_s$ and $\widehat{\alpha}_s$,

then categorize $[o]_s$ as belonging to the boundary set of χ.

(N2') If the probability of χ given $[o]_s$ is no greater than $\widehat{\beta}_s$,

then designate $[o]_s$ as belonging to the negative set of χ.

4. Development and application of novel algorithm

This section delves into the core principles and underlying structure of our novel algorithm. Subsequently, we subject it to a rigorous real-world test for air quality prediction. Through a comparative analysis with seven established algorithms, we evaluate the stability and accuracy of our approach. Finally, a comprehensive sensitivity analysis is conducted to determine the optimal parameter configurations for maximizing efficiency in practical applications.

4.1. Algorithm proposal

The following section provides a detailed overview of our innovative sorting algorithm, with the steps listed as follows.

- Step 1: Input the decision matrix.
- Step 2: Calculate the PLTSs matrix.
- Step 3: Calculate the attribute weights.
- Step 4: Calculate the innovative index for accessing group satisfaction.
- Step 5: Calculate the utility-aware values for each object.
- Step 6: Arrange all objects in descending order according to their utility-perceived value.

Next, we will introduce our classification algorithm in detail, and the programmed steps are detailed as follows.

Algorithm 1 Algorithm of the ranking Methods.

Input: A linguistic term evaluation table with g DMs whose weight vector is $\left(\varepsilon^{(1)}, \varepsilon^{(2)}, \ldots, \varepsilon^{(g)}\right)'$, the loss avoidance coefficient α, θ, the risk attitude coefficient λ, and gains and the utility pursuit coefficient ζ.
Output: The ranking results of all objects.
for $i = 1$ to n, $j = 1$ to m, $t = 1$ to g **do**
  The probabilistic linguistic term information matrix should be calculated in accordance with Section 3.1 of the article, , and normalized to $\mathfrak{R} = |L_{ij}(p)|_{n\times m}$.
end for
for $i = 1$ to n, $j = 1$ to m, $t = 1$ to g **do**
  Calculate the normalized attribute weights by constructing a linear programming model according to Section 3.2.
end for
for $i = 1$ to n, $j = 1$ to m **do**
  Calculate the innovative index for assessing group satisfaction $\varphi(L(p))$ in PLTSs defined by Section 3.3 to replace the utility value in PT.
end for
for $i = 1$ to n, $j = 1$ to m **do**
  The utility perception value is calculated for each object from the formula $F(o_i)$ Eq. (21).
end for
for $i = 1$ to n **do**
  Reorganize all objects in a descending hierarchy, prioritizing those with higher perceived utility values.
end for

Algorithm 2 Algorithm of the classification Methods.

Input: A linguistic term evaluation table with g DMs whose weight vector is $\left(\varepsilon^{(1)}, \varepsilon^{(2)}, \ldots, \varepsilon^{(g)}\right)'$, the loss avoidance coefficient α, θ, the risk attitude coefficient λ, and gains and the utility pursuit coefficient ζ.
Output: The classification results of all objects.
for $i = 1$ to n, $j = 1$ to m, $t = 1$ to g **do**
  The probabilistic linguistic term information matrix should be calculated in accordance with Section 3.1 of the article, , and normalized to $\mathfrak{R} = |L_{ij}(p)|_{n\times m}$.
end for
Determine the clustering number C according to [18], [19].
for $i = 1$ to n, $j = 1$ to m, $s = 1$ to C **do**
  Calculate the matrix Z by Section 3.4
end for
for $i = 1$ to n, $j = 1$ to m, $s = 1$ to C **do**
  Compute and updated membership matrix U according Fig. 2. If $\max_{1\le i\le c}\left\|a_i^{(k)} - a_i^{(k-1)}\right\| < \mathcal{E}$, the final membership matrix is determined;otherwise, let $a_i^{(k)} = a_i^{(k+1)}$ continue iterative calculation until $\max_{1\le i\le c}\left\|a_i^{(k)} - a_i^{(k-1)}\right\| < \mathcal{E}$ is satisfied.
end for
for $i = 1$ to n **do**
  Obtain the equivalence class $\Theta = \{[o]_1, [o]_2, \ldots, [o]_C\}$.
end for
for $i = 1$ to n, $s = 1$ to C **do**
  Calculate the conditional probability $Pr(\chi|[o]_s)$ under the equivalence class by Eq. (16).
end for
for $s = 1$ to C **do**
  Calculate the average utility perception value $\mathcal{F}([o]_s)$ by Eq. (23).
end for
for $s = 1$ to C **do**
  Compute the thresholds $\widehat{\alpha}_s$, $\widehat{\beta}_s$, and $\widehat{\eta}_s$ by Eq. (29).
end for
for $s = 1$ to C **do**
  Obtain classifications rules (P2') – (N2').
end for

- Step 1: Input Dataset.
- Step 2: Determine the C of FPCM.
- Step 3: Obtain the equivalence classes.
- Step 4: Calculate the conditional probability of each equivalence class.
- Step 5: Average utility perception values were calculated.
- Step 6: Calculate the three thresholds.
- Step 7: Obtain the classification results.

To make the algorithm's process clearer and more understandable, we've drawn a flowchart as shown in the Fig. 2. It illustrates the entire process from data input to final output, detailing each step and its logical connections.

4.2. Practical implementation and evaluation of algorithms in air quality

In this section, we explore the application of our method in addressing the complex air quality decision-making challenges in China. By adopting this approach, our objective is to furnish policymakers with effective decision support, thereby facilitating the resolution of this complex and pressing issue.

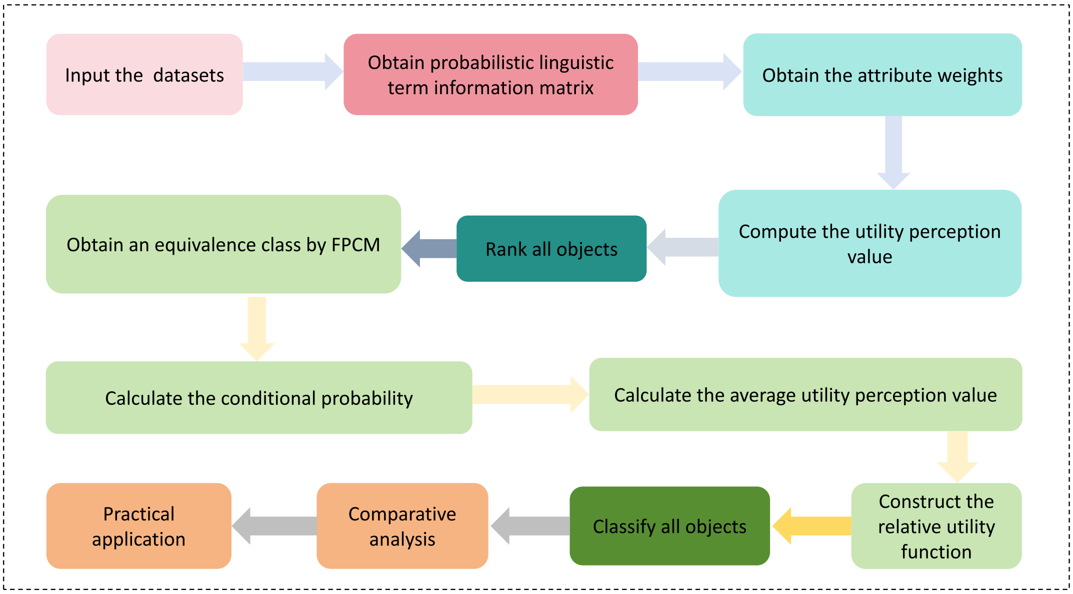

Fig. 2. Algorithm Flowchart.

4.2.1. Essential steps in data pre-processing for ranking and classification

Problem Description: China, renowned for its diverse and intricate terrain, faces a prominent societal challenge: air pollution, exacerbated by significant seasonal weather variations. The issue has been further compounded in recent years by rapid economic growth, a surge in motor vehicles, a swelling population, and continuous growth in manufacturing output. These factors have jointly intensified the degree of air pollution, posing a dire threat to public health.

To gain a deeper understanding of the specific impacts of six major air pollutants on China's air quality and evaluate the practical application effectiveness of our proposed method within a probabilistic framework, this paper meticulously gathered historical air quality data from various cities across China in 2021 (data source: https://www.aqistudy.cn/historydata). The data collection process adhered to strict standards, such as ensuring data integrity and consistency. We selected Beijing, Shanghai, and Guangzhou as representative cities, treating them as equally weighted decision-making units with a weight of 1/3 each.

Next, we transformed the numerical data of the Air Quality Index (AQI) in these three cities in 2021 into probabilistic linguistic term-based evaluation information. For example, if the AQI value of a certain day in Beijing is 80, and considering the range of AQI values and the defined linguistic levels ($\tau = 6$), this value falls within a specific interval that corresponds to a particular linguistic term, like s_2. This transformation was carried out following the rule: Assuming the numerical evaluation value under a given attribute is $y \in [m, M]$, where m represents the minimum value in the evaluation information and M the maximum. Taking the benefit attribute evaluation information as an example, when the evaluation value falls within $y \in [m, m + \frac{M-m}{\tau+1}]$, y converts to s_0; when $y \in (m + \frac{M-m}{\tau+1}, m + 2\times\frac{M-m}{\tau+1}]$, y converts to s_1, and so on. The conversion rule is reversed for the cost attribute evaluation information.

For the decision class in the evaluation information matrix, different processing methods were applied according to the data type. If the decision class information is linguistic data, it is transformed into $S = s_\gamma | \gamma = 0, 1, ..., \tau$ according to the aforementioned preprocessing. Then, $S_{\frac{\tau}{2}}$ is typically used as the cutoff point. For example, if an object's decision class is transformed into s_0, s_1, s_2 (when $\tau = 6$), the object belongs to state x; conversely, if it is transformed into s_4, s_5, s_6, the object belongs to state $\neg x$. If the decision information is numerical data, it is classified and processed based on the specific dataset. For instance, if the numerical decision data represents the number of days with excellent air quality in a month, we might set a threshold based on historical data analysis. If the number is above the threshold, the object is classified into one state, and if below, it is classified into another state.

Assuming the numerical evaluation value under a given attribute is $y \in [m, M]$, where m represents the minimum value in the evaluation information and M the maximum. Subsequently, we determined that there were $\tau + 1$ evaluation language levels. Taking the benefit attribute evaluation information as an example, when the evaluation value falls within $y\ in \left[m, m + \frac{M-m}{tau+1}\right]$, y converts to s_0.

The air quality dataset of China in 2021 comprises daily records spanning the entire year, encompassing 365 data points, collectively represented as the set $O = \{o_1, o_2, \ldots, o_{365}\}$. Each data point exhibits six pivotal attributes: $PM_{2.5}(a_1)$, $PM_{10}(a_2)$, $NO_2(a_3)$, $SO_2(a_4)$, $CO(a_5)$, and $O_3 - 8\,h(a_6)$. Furthermore, for datasets pertaining to three select cities, we have incorporated a crucial decision attribute—the air quality grade. To ensure an objective and precise classification of air quality grades, we have utilized the average Air Quality Index (AQI) as the benchmark. Concretely, an AQI within the range of [0,50] is deemed to indicate excellent air quality, whereas an AQI exceeding 50 is categorized as non-excellent. Based on this classification, we have redefined the quality grades into two distinct states: the state set $\Omega = \{\chi, \neg\chi\}$. Notably, in this context, we have deliberately redefined the meaning of state χ. Specifi-

Table 2
Comparison of Ranking Algorithms.

Methods	Data type	Behavioral psychology	Weights	Conditional probability	Thresholds	Ranking	Classification
M0	PLTSs	yes	Objective	Objective	Objective	yes	yes
M1	PLTSs	yes	Objective	Objective	Objective	yes	yes
M2	HOPSIS	yes	Subjective	no	no	yes	no
M3	HOPSIS	yes	Subjective	no	no	yes	no
M4	PLTSs	no	Objective	no	no	yes	no
M5	HFSs	no	Objective	no	no	yes	no
M6	HFSs	yes	Objective	Objective	no	yes	no
M7	HFSs	yes	Objective	Objective	no	yes	no
M8	HFSs	no	Objective	no	no	yes	no

cally, the "good" state χ represents objects with non-excellent air quality, while the "bad" state $\neg\chi$ corresponds to objects exhibiting excellent air quality. Consequently, all six attributes are considered beneficial for analysis.

For the decision class in the evaluation information matrix, if the decision class information is linguistic data, it is transformed into $S = \{s_\gamma | \gamma = 0, 1, \ldots, \tau\}$ according to the aforementioned preprocessing. Then, $s_{\frac{\tau}{2}}$ is typically used as the cutoff point. If the object's decision class is transformed into $s_0, s_1, \ldots, s_{\frac{\tau}{2}}$, the object belongs to a state χ. Conversely, when the object's decision class is transformed into $s_{(\frac{\tau}{2}+1)}, s_{(\frac{\tau}{2}+2)}, \ldots, s_\tau$, the object belongs to a state $\neg\chi$. If the decision information is numerical data, it is classified and processed based on the specific dataset.

4.2.2. Comprehensive study on the impact of ranking algorithms

In this section, we provide a comprehensive overview of the PT and the TWD employed in this study. By integrating the findings from Section 3.2, we determine the attribute weights as follows: $0.1465, 0.1571, 0.1589, 0.1353, 0.1737$, and 0.2283. In accordance with Algorithm 1 and the sensitivity analysis in Section 4.3, we establish the loss aversion coefficient α at 0.88, assign a value of 0.68 to θ, and set the risk attitude coefficient λ at 2.25. The results derived from these parameter settings are clearly illustrated in Fig. 4. These outcomes validate the accuracy of our algorithm and facilitate a comparative analysis with seven other algorithms.

Firstly, we have designated our independently developed algorithm as M0. Additionally, the algorithm introduced by Zhu et al. has been designated as M1 [20], the algorithm introduced by Zhang et al. as M2 [17], the algorithm proposed by another group led by Zhang et al. as M3 [21], and the algorithm proposed by Wu et al. as M4 [12]. Furthermore, the algorithm proposed by Liu et al. is designated as M5 [22], the algorithm proposed by another team as M6 [23], and the algorithm introduced by yet another team as M7 [24]. The method introduced by Liu et al. is designated as M8 [25]. Table 2 presents a compilation of the fundamental information pairs pertaining to the aforementioned distinct algorithms. We now proceed with a rigorous, prudent, and rational comparative analysis of each methodology, as follows:

(1) In the probabilistic linguistic context, notable disparities emerged between M1 and M4. Specifically, M1 incorporates a profound consideration of decision-makers' regret, thereby introducing a more humane dimension into its decision-making framework. Conversely, M4 lacks an explicit focus on decision-maker psychology. To illustrate the impact of these differences, consider a real - world decision - making scenario of a company choosing an investment project in an emerging industry. When using M1, the decision - maker, driven by the fear of regret in case of investment failure, will conduct a more comprehensive risk assessment. For example, they will analyze the intense market competition in the emerging industry, where new competitors may enter the market rapidly, and the high speed of technological updates that could render the invested technology obsolete in a short time. This comprehensive analysis helps the decision - maker avoid impulsive investment and potential losses. In contrast, M4, which does not consider decision - maker psychology, may rely solely on objective project data. It might focus only on the projected financial returns and overlook intangible risks such as the potential damage to the company's reputation if the project fails. For instance, if the investment fails due to unforeseen market changes, the company's image among investors and customers could be severely damaged, leading to long - term negative impacts on its business operations. This oversight can result in less rational decision - making. The comparative outcomes are comprehensively presented in Fig. 5 and Fig. 8.
(2) Furthermore, both M2 and M3 share a common emphasis on decision-maker psychology, albeit without incorporating external environmental factors or other variables. The decision outcomes of these methodologies are contrasted in Fig. 6 and Fig. 7, allowing for a comprehensive understanding of the similarities and differences in their psychological approaches.
(3) Additionally, within the context of ambiguous hesitation, M4 through M8 exhibit distinct characteristics. Notably, M5 solely accounts for the fuzzy nature of hesitation, excluding other complex considerations. On the other hand, M8 integrates a bi-directional projection algorithm, providing a more precise analytical approach to the decision-making process. Furthermore, M6 and M7 incorporate psychoanalysis into their decision-making frameworks, thereby enhancing the realism of their outcomes. The comprehensive comparison of decision outcomes is detailed in Fig. 9, Fig. 10, Fig. 11 and Fig. 12.

To further validate the practicality of our proposed algorithm, we have consequently opted to utilize the Spearman rank correlation coefficient (SRCC) [26] within the realm of statistics. This coefficient serves to assess the relationship between the two variables, specifically the similarity ranking between the two sets of ranking results. Based on these ranking outcomes, we have provided the following definition of SRCC.

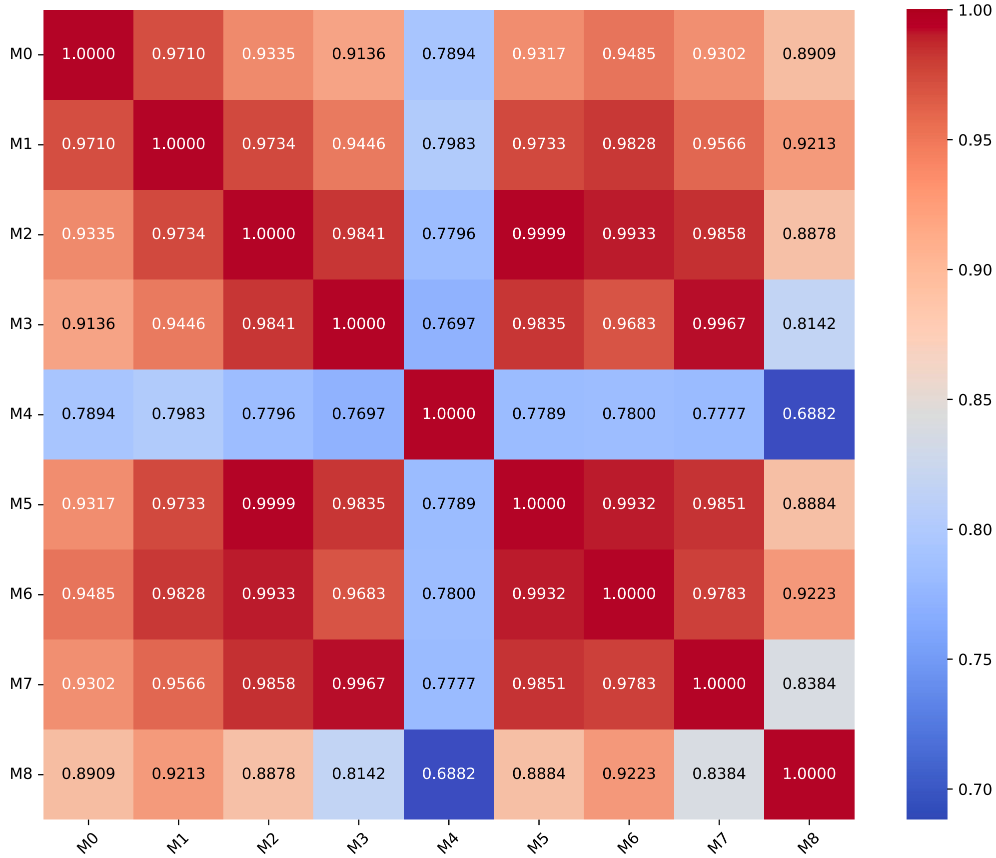

Fig. 3. The Spearman Correlation Coefficients between Matrices. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

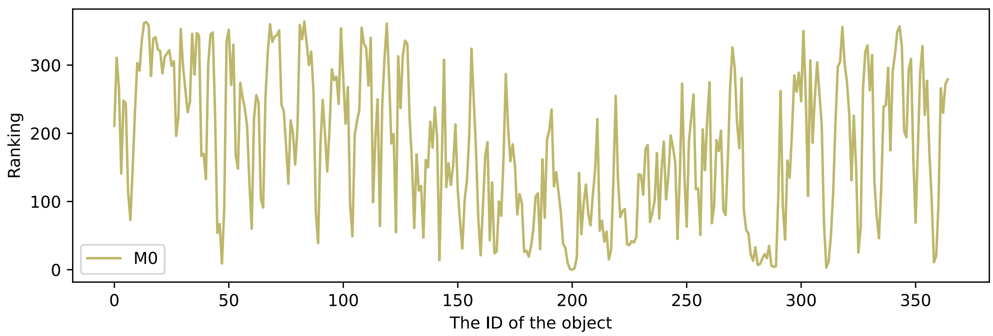

Fig. 4. The Ranking Results.

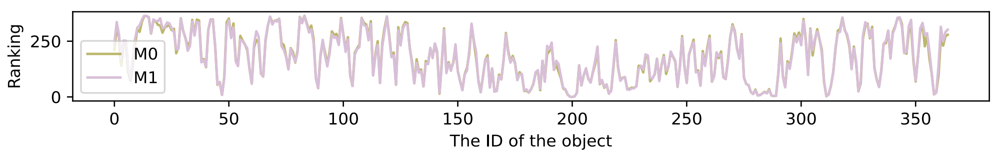

Fig. 5. Comparison with M1.

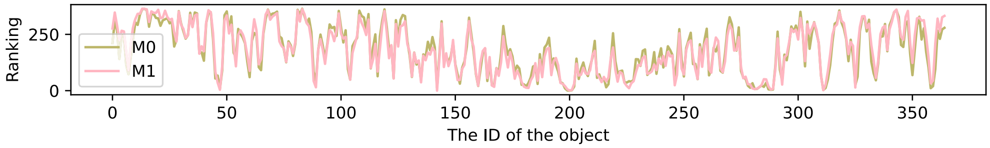

Fig. 6. Comparison with M2.

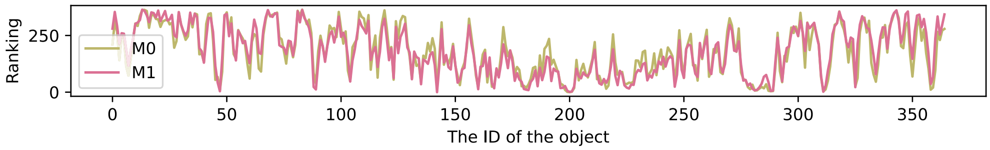

Fig. 7. Comparison with M3.

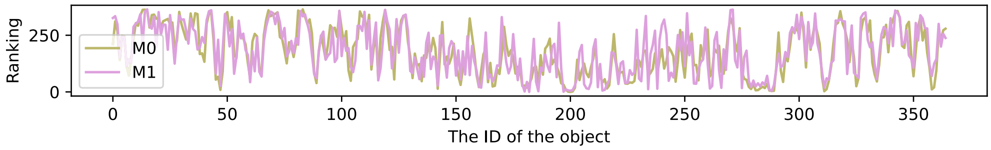

Fig. 8. Comparison with M4.

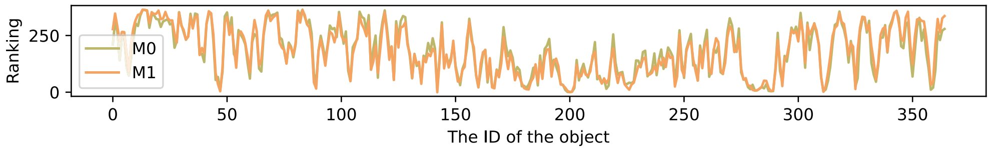

Fig. 9. Comparison with M5.

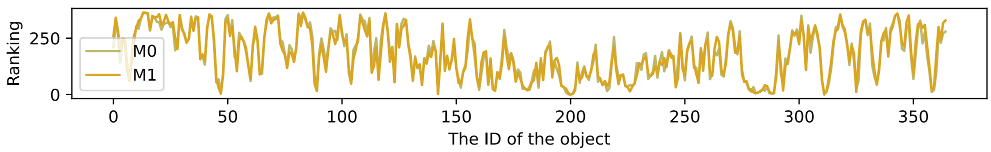

Fig. 10. Comparison with M6.

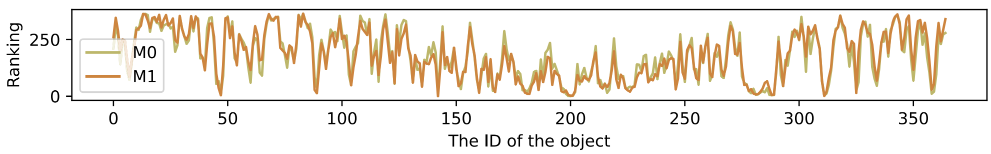

Fig. 11. Comparison with M7.

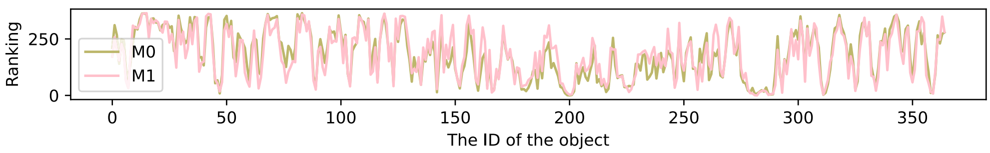

Fig. 12. Comparison with M8.

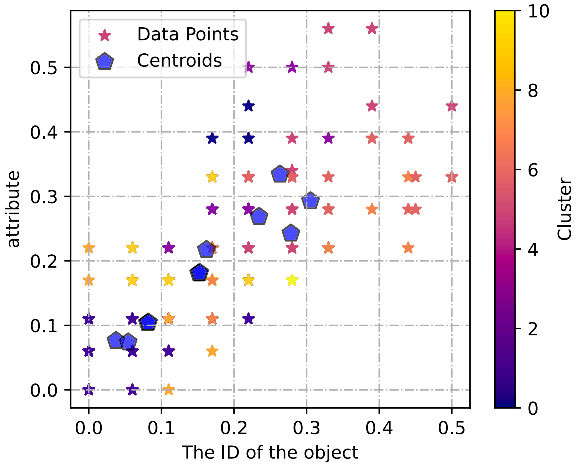

Fig. 13. The FPCM Clustering Results.

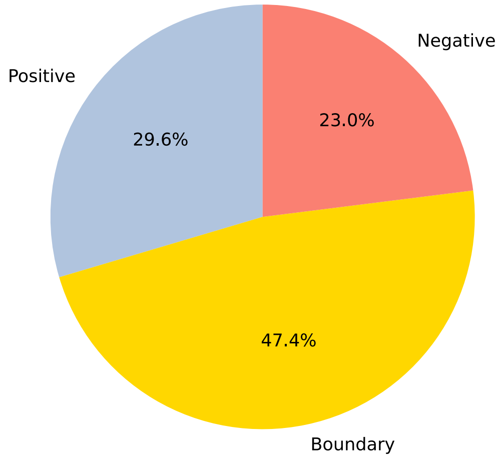

Fig. 14. Distribution of positive, Boundary and Negative samples of FPCM.

The Spearman's Rank Correlation Coefficient (SRCC) is a statistical measure that quantifies the strength and direction of a monotonic relationship between two sorting methods when applied to a set of objects. In our context, let's assume we have a set of objects O containing n unique elements, and two different sorting methods, M' and M'', which each produce a unique ordered sequence of these objects.

Each method ranks the objects from the most preferred to the least preferred, resulting in two distinct sequences. Within these sequences, each object occupies a specific position, and the positions of interest to us are those of o_{x_e} and o_{y_f} in M' and M'' respectively, denoted by e and f.

The SRCC formula captures the similarity between these two rankings by summing the squared differences in the positions of corresponding objects and normalizing this sum. The resulting value ranges from -1 to 1, with 1 indicating a perfect positive correlation (identical rankings), -1 indicating a perfect negative correlation (opposite rankings), and 0 indicating no correlation (random rankings). In our case, the SRCC is calculated using the formula given in equation (29). This formula subtracts from 1 the value obtained by multiplying 6 with the sum of squared position differences between the two rankings, divided by the product of the number of objects and the square of the number of objects minus 1. The result is a measure of how closely the two sorting methods agree in their ordering of the objects in the set O.

The Spearman's coefficient results between the methods have been computed and are presented in Fig. 3. Through this series of experimental verification and data analysis, we fully prove the effectiveness of M0 algorithm and its advantages in practical application.

4.2.3. Optimal classification strategies for air quality problems

It is imperative to accurately determine the cluster number, denoted as C. Initially, we adhere to the standard heuristic approach in the field, setting C approximately equal to the square root of the total number of objects, n. Following this, we employ the FPCM

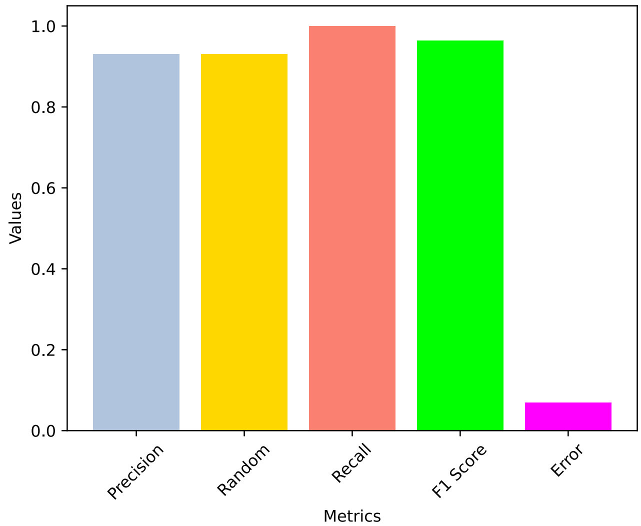

Fig. 15. Evaluation Metrics for Classification of FPCM.

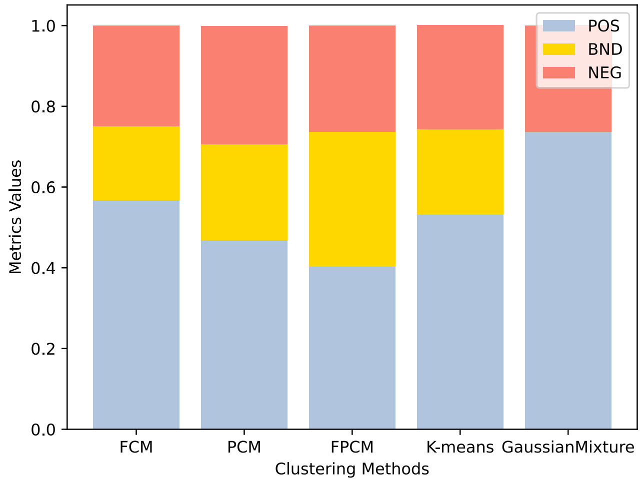

Fig. 16. Comparison of Clustering Methods.

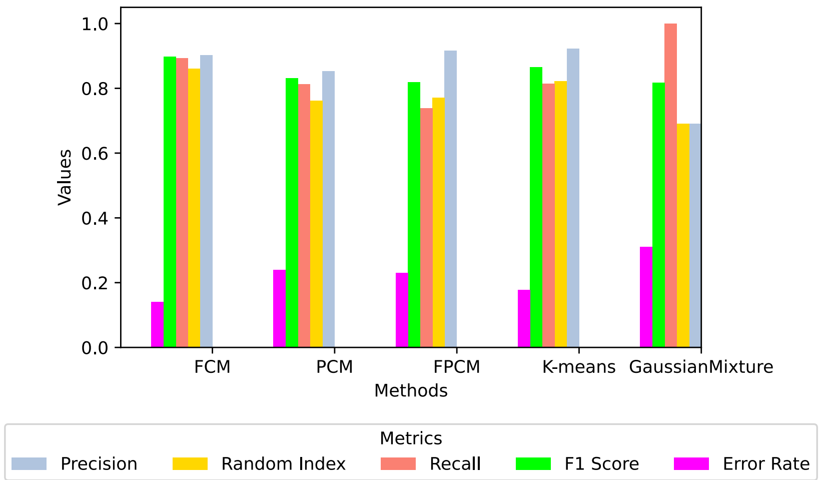

Fig. 17. Comparison of Clustering Methods.

Table 3
Comparison of Clustering Algorithms.

Algorithm	Advantages	Disadvantages
FCM	Handles fuzziness and uncertainty	High complexity for large datasets
FCM	Flexibility in handling uncertainty	High complexity, requires parameter tuning
FPCM	Combines fuzziness and possibility	Highest complexity, more parameter tuning
K-means	Simple, efficient for large datasets	Sensitive to initial cluster centers
GMM	Handles complex distributions	High complexity for high-dimensional data

algorithm for clustering analysis, and the resultant clusters are depicted in Fig. 13. Subsequently, we adjust the parameter ζ to 0.9 and obtain the proportional distribution of positive, negative, and boundary domains, as illustrated in Fig. 14.

In the real - world air quality monitoring process, air quality data is amassed continuously from numerous monitoring stations. The FPCM algorithm, with its superior capability to precisely distinguish different categories within the air quality data, can detect minute variations in pollutant concentrations. It has the capacity to promptly identify abnormal changes, which in turn aids the environmental protection department in swiftly pinpointing potential pollution sources. As a result, a more timely response can be initiated.

To further validate the clustering performance, we introduce a comprehensive set of evaluation metrics, including error rate, recall rate, precision, and F1 score. These metrics are defined in the following section, and their assessment outcomes are presented in Fig. 15, providing a detailed analysis of the clustering effect.

$$\begin{aligned}
\text{Error rate} &= \frac{n_{\chi\to NEG(\chi)} + n_{\neg\chi\to POS(\chi)}}{n_{POS(\chi)} + n_{NEG}(\chi)} \times 100\% \\
\text{Precision} &= \frac{n_{\chi\to POS(\chi)}}{n_{\chi\to POS(\chi)} + n_{\neg\chi\to POS(\chi)}} \times 100\% \\
\text{Rand index} &= \frac{n_{\chi\to POS(\chi)} + n_{\neg\chi\to NEG(\chi)}}{n_{\chi} + n_{\neg\chi}} \times 100\% \\
\text{Recall} &= \frac{n_{\chi\to POS(\chi)}}{n_{\chi\to POS(\chi)} + n_{\chi\to NEG(\chi)}} \times 100\% \\
\text{F1} &= \frac{2 * Precision * Recall}{Precision + Recall} \times 100\%
\end{aligned} \tag{30}$$

A high level of precision indicates that the air quality management measures devised according to the classification outcomes of this algorithm have a greater probability of attaining the desired results. Leveraging these results, both the government and environmental protection agencies are enabled to allocate resources in a more efficient manner. This not only mitigates the risk of making erroneous decisions but also cuts down on the costs associated with air quality governance. In this region, the data regarding air quality is characterized by a high degree of uncertainty, primarily attributed to the intricate interplay among multiple pollution sources. However, the FPCM algorithm proves to be highly effective in managing such uncertainty. It has the capacity to precisely demarcate regions with varying degrees of pollution. This, in turn, provides policymakers with the necessary support to formulate more targeted and refined environmental policies.

Finally, we used FPCM as M0 and compared it with the other four methods. We used FCM [27] algorithm as M1, PCM [28] as M2, K-means [29] as M3, and Gaussian mixture model [30] as M4. The pairs of the five clustering algorithms are shown in Table 3. The comparison of positive domain, negative domain and boundary domain is shown in Fig. 16, and the comparison of accuracy and error rate is shown in Fig. 17.

After examining the color regions and distributions within the graph, it becomes evident that the FPCM exhibits superior performance in terms of both the accuracy and stability of its clustering results. Conversely, the remaining four models exhibit certain biases and uncertainties during the clustering process, ultimately leading to relatively lower accuracy and reliability in their outcomes.

Furthermore, a comparative analysis of accuracy and error rate indicators is presented in Fig. 17. This figure clearly demonstrates that the FPCM outperforms the other four models across all performance metrics. Specifically, the FPCM exhibits remarkable clustering accuracy, effectively identifying distinct categories of data points with precision. Additionally, its relatively low error rate suggests a reduced incidence of misjudgments and errors during the clustering process.

To further buttress our position, we delve into the underlying principles and applicable scenarios of each model during clustering. The FCM algorithm enhances clustering outcomes by incorporating fuzzy factors, but it may encounter computational challenges when dealing with large-scale datasets. The PCM algorithm mitigates the impact of noise points on clustering outcomes by introducing the likelihood concept, albeit with the potential to obscure cluster boundaries in certain instances. The K-means algorithm offers simplicity and computational efficiency, yet it may falter when confronted with data exhibiting significant non-convex shapes or size variations. Gaussian mixed models are capable of handling more intricate data distributions and shapes, yet they can be complicated in terms of parameter estimation and model selection.

In summary, our comparative analysis of various clustering methods underscores the superiority of FPCM as an effective clustering approach. Within the context of this paper, FPCM emerges as the preferred choice, outperforming the other four models in terms of clustering accuracy and stability.

Table 4
$\lambda = 1.25$: Comparison of Spearman and Average Coefficients of M0 - M8 Algorithms for Different α, θ Combinations.

α	θ	M0	M1	M2	M3	M4	M5	M6	M7	M8	average
0.98	0.98	1.0000	0.9998	0.9724	0.9442	0.7991	0.9722	0.9822	0.9566	0.9208	0.9496
0.88	0.98	1.0000	0.9967	0.9660	0.9401	0.7998	0.9653	0.9774	0.9537	0.9172	0.9462
0.78	0.98	1.0000	0.9889	0.9553	0.9324	0.7979	0.9541	0.9681	0.9472	0.9078	0.9390
0.68	0.98	1.0000	0.9772	0.9407	0.9200	0.7925	0.9390	0.9548	0.9360	0.8965	0.9285
0.58	0.98	1.0000	0.9633	0.9244	0.9058	0.7858	0.9223	0.9398	0.9228	0.8828	0.9163
0.48	0.98	1.0000	0.9504	0.9102	0.8933	0.7776	0.9079	0.9263	0.8109	0.8700	0.9051
0.98	0.88	1.0000	0.9993	0.9743	0.9440	0.7963	0.9745	0.9833	0.9556	0.9222	0.9499
0.78	0.88	1.0000	0.9997	0.9721	0.9438	0.7986	0.9718	0.9821	0.9563	0.9214	0.9456
0.68	0.88	1.0000	0.9876	0.9537	0.9307	0.7972	0.9524	0.9667	0.9458	0.9071	0.9379
0.58	0.88	1.0000	0.9748	0.9378	0.9174	0.7907	0.9361	0.9523	0.9336	0.8946	0.9263
0.48	0.88	1.0000	0.9602	0.9208	0.9023	0.7841	0.9187	0.9365	0.9195	0.8802	0.9135
0.98	0.78	1.0000	0.9965	0.9742	0.9418	0.7920	0.9747	0.9825	0.9527	0.9224	0.9485
0.88	0.78	1.0000	0.9992	0.9743	0.9438	0.7963	0.9744	0.9834	0.9555	0.9227	0.9499
0.78	0.78	1.0000	0.9995	0.9717	0.9433	0.7989	0.9714	0.9821	0.9561	0.9220	0.9494
0.68	0.78	1.0000	0.9954	0.9641	0.9382	0.7990	0.9633	0.9760	0.9521	0.9168	0.9449
0.58	0.78	1.0000	0.9857	0.9509	0.9281	0.7962	0.9496	0.9645	0.9435	0.9056	0.9360
0.48	0.78	1.0000	0.9720	0.9347	0.9146	0.7900	0.9329	0.9495	0.9310	0.8920	0.9240
0.98	0.68	1.0000	0.9932	0.9723	0.9382	0.7869	0.9732	0.9804	0.9488	0.9219	0.9460
0.88	0.68	1.0000	0.9963	0.9740	0.9415	0.7918	0.9746	0.9823	0.9524	0.92267	**0.9501**
0.78	0.68	1.0000	0.9988	0.9702	0.9428	0.7999	0.9698	0.9810	0.9559	0.9209	0.9483
0.68	0.68	1.0000	0.9991	0.9740	0.9434	0.7964	0.9742	0.9833	0.9552	0.9229	0.9498
0.58	0.68	1.0000	0.9992	0.9711	0.9428	0.7991	0.9708	0.9819	0.9558	0.9224	0.9492
0.48	0.68	1.0000	0.9947	0.9632	0.9375	0.7989	0.9642	0.9753	0.9516	0.9162	0.9443
0.98	0.58	1.0000	0.9837	0.9486	0.9259	0.7963	0.9472	0.9624	0.9416	0.9043	0.9343
0.88	0.58	1.0000	0.9897	0.9702	0.9349	0.7836	0.9712	0.9777	0.9451	0.9199	0.9435
0.78	0.58	1.0000	0.9961	0.9737	0.9410	0.7913	0.9743	0.9822	0.9520	0.9231	0.9481
0.68	0.58	1.0000	0.9988	0.9737	0.9429	0.7965	0.9739	0.9833	0.9549	0.9235	0.9497
0.58	0.58	1.0000	0.9988	0.9706	0.9424	0.7996	0.9702	0.9815	0.9555	0.9222	0.9489
0.48	0.58	1.0000	0.9933	0.9614	0.9358	0.7982	0.9605	0.9738	0.9501	0.9153	0.9431

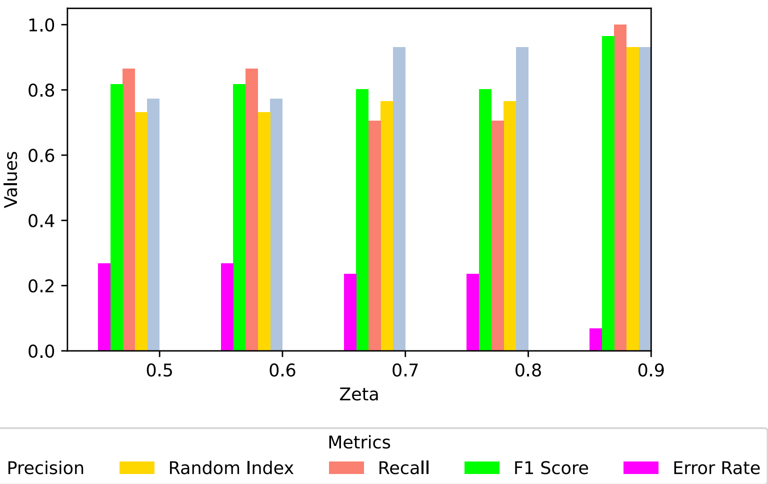

Fig. 18. The classification results vary depending on the value of ζ.

4.3. Sensitivity analysis

During the sensitivity analysis of sequencing results, it is imperative to conduct a thorough transformation analysis of the parameters λ, α, and θ. Based on prior knowledge, we recognize that the permissible value range for λ is subject to the condition $\lambda > 1$. Additionally, the values of the parameters α and θ are confined to the closed interval between 0 and 1. Consequently, in our subsequent sensitivity analysis, we have selected three representative values for λ, specifically $\lambda = 1.25$, $\lambda = 2.25$, and $\lambda = 3.25$, in order to comprehensively assess their impact on the ranking results. However, upon examining the influence of α and θ, we observed that the numerical disparities in the resulting utility pursuit value function were minimal when both parameters were less than 0.5, potentially introducing significant errors. Therefore, to enhance the precision of our analysis, we have chosen a value of 0.1 within the range of 0.48 to 0.98 for α and similarly, a value of 0.1 within the range of 0.58 to 0.98 for θ. Finally, to quantify the sensitivity of the ranking results, we have calculated the average value of the Spearman's coefficient using various alternative methods for each parameter set. The results obtained are presented in Table 4, Table 5 and Table 6. In the process of conducting a thorough cluster

Table 5
$\lambda = 2.25$: Comparison of Spearman and Average Coefficients of M0 - M8 Algorithms for Different α, θ Combinations.

α	θ	M0	M1	M2	M3	M4	M5	M6	M7	M8	average
0.98	0.98	1.0000	0.9947	0.9636	0.9390	0.7998	0.9627	0.9750	0.9528	0.9137	0.9446
0.88	0.98	1.0000	0.9860	0.9514	0.9294	0.7967	0.9500	0.9645	0.9445	0.9046	0.9364
0.78	0.98	1.0000	0.9740	0.9368	0.9168	0.7905	0.9350	0.9512	0.9331	0.8930	0.9255
0.68	0.98	1.0000	0.9609	0.9217	0.9033	0.7845	0.9196	0.9373	0.9205	0.8804	0.9142
0.58	0.98	1.0000	0.9490	0.9068	0.8921	0.7769	0.9063	0.9247	0.9096	0.8683	0.9039
0.48	0.98	1.0000	0.9387	0.8974	0.8817	0.7708	0.8948	0.9138	0.8987	0.8592	0.8951
0.98	0.88	1.0000	0.9991	0.9706	0.9432	0.7983	0.9702	0.9810	0.9561	0.9201	0.9487
0.88	0.88	1.0000	0.9942	0.9627	0.9380	0.7996	0.9618	0.9744	0.9519	0.9138	0.9440
0.78	0.88	1.0000	0.9848	0.9498	0.9277	0.7958	0.9484	0.9632	0.9431	0.9041	0.9352
0.68	0.88	1.0000	0.9723	0.9349	0.9151	0.7898	0.9331	0.9496	0.9315	0.8917	0.9242
0.58	0.88	1.0000	0.9587	0.9190	0.9008	0.7830	0.9169	0.9348	0.9181	0.8785	0.9122
0.48	0.88	1.0000	0.9462	0.9057	0.8843	0.7752	0.9032	0.9348	0.9218	0.9071	0.8556
0.98	0.78	1.0000	0.9998	0.9738	0.9443	0.7977	0.9738	0.9832	0.9563	0.9221	**0.9501**
0.88	0.78	1.0000	0.9989	0.9703	0.9429	0.7989	0.9699	0.9810	0.9559	0.9206	0.9487
0.78	0.78	1.0000	0.9937	0.9619	0.9370	0.7991	0.9609	0.9739	0.9512	0.9140	0.9435
0.68	0.78	1.0000	0.9835	0.9483	0.9262	0.7952	0.9468	0.9620	0.9418	0.9032	0.9341
0.58	0.78	1.0000	0.9835	0.9483	0.9262	0.7952	0.9468	0.9620	0.9418	0.9032	0.9341
0.48	0.78	1.0000	0.9563	0.9164	0.8984	0.7812	0.9142	0.9324	0.9159	0.8763	0.9101
0.98	0.68	1.0000	0.9975	0.9744	0.9462	0.7938	0.9748	0.9830	0.9538	0.9226	0.9491
0.88	**0.68**	**1.0000**	**0.9997**	**0.9738**	**0.9441**	**0.7974**	**0.9739**	**0.9833**	**0.9561**	**0.9226**	**0.9501**
0.78	0.68	1.0000	0.9988	0.9702	0.9428	0.7999	0.9698	0.9810	0.9559	0.9209	0.9488
0.68	0.68	1.0000	0.9932	0.9610	0.9362	0.7983	0.9601	0.9733	0.9505	0.9138	0.9429
0.58	0.68	1.0000	0.9823	0.9468	0.9247	0.7943	0.9454	0.9608	0.9405	0.9025	0.9341
0.48	0.68	1.0000	0.9682	0.9303	0.9108	0.7883	0.9283	0.9454	0.9375	0.8882	0.9207
0.98	0.58	1.0000	0.9941	0.9729	0.9392	0.7884	0.9737	0.9810	0.9500	0.9221	0.9461
0.88	0.58	1.0000	0.9972	0.9741	0.9420	0.7930	0.9745	0.9828	0.9533	0.9230	0.9488
0.78	0.58	1.0000	0.9995	0.9737	0.9438	0.7974	0.9738	0.9833	0.9559	0.9229	0.9500
0.68	0.58	1.0000	0.9985	0.9698	0.9422	0.8000	0.9694	0.9809	0.9554	0.9214	0.9486
0.58	0.58	1.0000	0.9924	0.9599	0.9350	0.7981	0.9590	0.9725	0.9495	0.9134	0.9421
0.48	0.58	1.0000	0.9806	0.9448	0.9230	0.7930	0.9433	0.9590	0.9389	0.9011	0.9315

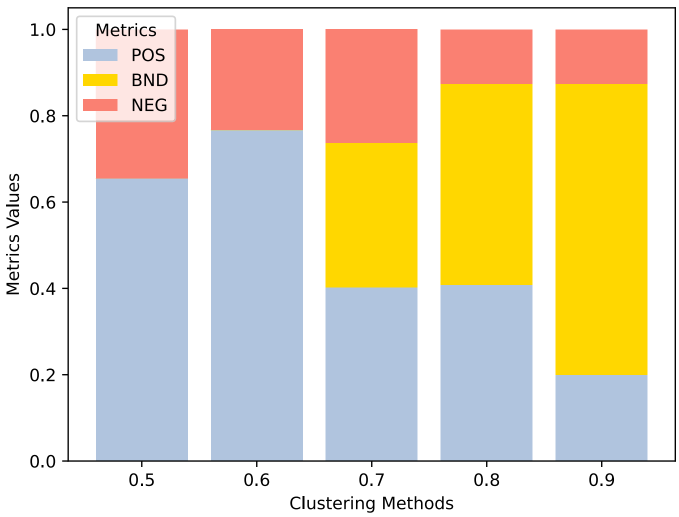

Fig. 19. The classification results vary depending on the value of ζ.

analysis, a meticulous examination of the core parameter ζ within the three decision frameworks was conducted. This parameter holds a pivotal position in the three decision theories, and its numerical assignment holds a direct correlation with the delineation of decision boundaries as well as the precision of the ultimate decision outcome. Drawing from the available data, it is established that the value of ζ fluctuates between 0.5 and 1. Hence, a range of values falling within this spectrum were chosen, and a comprehensive assessment of crucial parameters, including error rate and accuracy, was carried out across these values. The specific outcomes are illustrated in Fig. 18 and Fig. 19. Through a series of intricate and meticulous computational and analytical endeavors, we successfully derived comparative results pertaining to the boundary domains, which were visually represented in graphical format as depicted in Fig. 19. This graphical representation clearly depicts the variations in the boundary domain across different values of ζ. Furthermore, an in-depth comparative analysis of key metrics such as accuracy was also conducted, and the comparison results were graphically presented in Fig. 1. Referring to Fig. 18, it is evident that both decision accuracy and error rate exhibit a noticeable fluctuation

Table 6
$\lambda = 3.25$: Comparison of Spearman and Average Coefficients of M0 - M8 Algorithms for Different α, θ Combinations.

α	θ	M0	M1	M2	M3	M4	M5	M6	M7	M8	average
0.98	0.98	1.0000	0.9933	0.9614	0.9358	0.7982	0.9605	0.9738	0.9501	0.9153	0.9431
0.88	0.98	1.0000	0.9878	0.9537	0.9314	0.7974	0.9524	0.9665	0.9463	0.9062	0.9379
0.78	0.98	1.0000	0.9640	0.9253	0.9068	0.7857	0.9233	0.9406	0.9237	0.8833	0.9169
0.68	0.98	1.0000	0.9515	0.9114	0.8945	0.7788	0.9091	0.9274	0.9120	0.8709	0.9061
0.58	0.98	1.0000	0.9412	0.9000	0.8840	0.7722	0.8975	0.9164	0.9019	0.8615	0.8971
0.48	0.98	1.0000	0.9342	0.8924	0.8772	0.7683	0.8898	0.9091	0.8954	0.8550	0.8912
0.98	0.88	1.0000	0.9954	0.9644	0.9394	0.7998	0.9636	0.9758	0.9531	0.9149	0.9451
0.88	0.88	1.0000	0.9871	0.9528	0.9305	0.7972	0.9515	0.9658	0.9455	0.9060	0.9373
0.78	0.88	1.0000	0.9754	0.9385	0.9183	0.7913	0.9368	0.9528	0.9344	0.8945	0.9268
0.68	0.88	1.0000	0.9623	0.9232	0.9048	0.7851	0.9212	0.9387	0.9219	0.8818	0.9154
0.58	0.88	1.0000	0.9502	0.9099	0.8932	0.7774	0.9076	0.9259	0.9108	0.8694	0.9049
0.48	0.88	1.0000	0.9395	0.8982	0.8824	0.7712	0.8956	0.9146	0.9004	0.8598	0.8957
0.98	0.78	1.0000	0.9994	0.9713	0.9436	0.7989	0.9709	0.9816	0.9563	0.9207	0.9491
0.88	0.78	1.0000	0.9951	0.9639	0.9387	0.7994	0.9630	0.9755	0.9526	0.9153	0.9448
0.78	0.78	1.0000	0.9863	0.9518	0.9293	0.7969	0.9505	0.9651	0.9444	0.9057	0.9366
0.68	0.78	1.0000	0.9741	0.9370	0.9169	0.7906	0.9352	0.9515	0.9331	0.8935	0.9257
0.58	0.78	1.0000	0.9602	0.9208	0.9024	0.7840	0.9187	0.9365	0.9196	0.8801	0.9135
0.48	0.78	1.0000	0.9481	0.9076	0.8911	0.7765	0.9052	0.9238	0.9087	0.8675	0.9031
0.98	0.68	1.0000	0.9996	0.9741	0.9442	0.7970	0.9742	0.9834	0.9560	0.9224	0.9500
0.88	0.68	1.0000	0.9993	0.9711	0.9431	0.7989	0.9707	0.9816	0.9560	0.9214	0.9491
0.78	0.68	1.0000	0.9949	0.9635	0.9380	0.7988	0.9627	0.9754	0.9520	0.9159	0.9445
0.68	0.68	1.0000	0.9854	0.9507	0.9282	0.7964	0.9494	0.9642	0.9436	0.9050	0.9358
0.58	0.68	1.0000	0.9727	0.9355	0.9155	0.7900	0.9337	0.9502	0.9319	0.8924	0.9246
0.48	0.68	1.0000	0.9538	0.9186	0.9003	0.7824	0.9164	0.9345	0.9177	0.8783	0.9118
0.98	0.58	1.0000	0.9968	0.9742	0.9419	0.7924	0.9747	0.9826	0.9530	0.9227	0.9487
0.88	0.58	1.0000	0.9993	0.9741	0.9439	0.7965	0.9742	0.9834	0.9557	0.9228	0.9500
0.78	0.58	1.0000	0.9992	0.9710	0.9429	0.7990	0.9707	0.9817	0.9559	0.9220	0.9491
0.68	0.58	1.0000	0.9948	0.9633	0.9376	0.7987	0.9625	0.9753	0.9517	0.9162	0.9444
0.58	0.58	1.0000	0.9846	0.9496	0.9270	0.7957	0.9483	0.9633	0.9425	0.9049	0.9351
0.48	0.58	1.0000	0.9710	0.9335	0.9136	0.7894	0.9317	0.9485	0.9302	0.8909	0.9232

trend in accordance with changes in the value of ζ. After meticulous comparison and rigorous analysis, we ultimately arrived at the conclusion that the decision outcomes were optimized when the value of ζ was set to 0.9.

Upon conducting an in-depth analysis of the aforementioned table data and image data, we have concluded that setting the parameters λ to 2.25, α to 0.88, θ to 0.68, and ζ to 0.9 will yield the optimal experimental results.

5. Conclusion

Within the probabilistic linguistic framework, the exploration of TWD represents a noteworthy theoretical leap in TWD research. By employing PLTSs as the core evaluative metric, this study adeptly incorporates the qualitative and hesitant preferences of decision-makers. The key contributions of this work are as follows:

(1) Theoretically, this study delves into the psychological characteristics of decision-making subjects, innovatively integrating the emotions of regret and jubilation into the TWD model. This integration, especially in tandem with the unique features of PLTSs, offers a richer and deeper understanding of decision analysis. The approach taken here not only enhances the comprehensiveness of the analysis but also adds a new dimension to our understanding of decision-making processes.

(2) To reconcile the disparities between targets and NISs, and to clarify the ambiguity surrounding attributes, a novel method for computing attribute weights has been devised. Tailored specifically for the PLTSs environment, this approach aims to enhance the accuracy of decision-making processes.

(3) By harnessing the FPCM algorithm for conditional probability computation and developing a novel relative utility function rooted in prospect theory, we introduce the PL-TWDP method, tailored to address practical challenges. Experimental results underscore the method's superior classification capabilities, evidence of its exceptional practical performance.

(4) Addressing the scarcity of authentic probabilistic linguistic datasets, this study adopts a novel method by transforming numerical datasets into linguistic evaluation tables using average range partitioning. This smooth integration into the necessary probabilistic linguistic evaluation information tables not only overcomes the obstacles encountered in the practical application of the method but also improves the usability and practicality of the data. Furthermore, the study validates the feasibility and effectiveness of this conversion strategy through rigorous empirical analysis and case studies, thereby providing valuable data support and theoretical reference for related research fields.

When considering multiple decision-makers, scenarios involving significant opinion disparities among them remain underexplored. Future research endeavors should delve deeper into strategies aimed at reconciling the opinions of decision-makers and developing effective consensus models. Moreover, in the field of fuzzy large-scale group decision-making, the combination of TWD with PLTS not only aids in overcoming challenges related to large-scale consensus but also holds substantial research potential in proposing fuzzy preference clustering methods. These methods can reduce clustering dimensionality and effectively manage conflicts within large groups.

CRediT authorship contribution statement

Xiaoyan Zhang: Supervision, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Feiyu Liu:** Writing – original draft, Visualization, Validation, Software, Data curation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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